School of Natural Sciences
Department of Mathematics
Assignment 1 (Integral Equations), Spring 2020
Solution Key

Q.1 Classify the following integral equations as Volterra, Fredholm, linear, non-linear, homogeneous, non-homogeneous, singular, non-singular, first kind, and second kind.

a. $\int_a^x (x^2t - xt^2)y(t)dt = f(x)$

Sol. Volterra Equation, Linear, Non-homogeneous, Non-singular, First Kind.

b. $\int_0^1 \frac{\sqrt{f(t)}}{x-t}dt = 1 - x + f(x)$

Sol. Fredholm Equation, Non-linear, Non-homogeneous, Singular, Second Kind.

c. $\lambda \int_0^{+\infty} e^{-st} f(t)dt = f(s), \qquad \lambda \in \mathbb{R}, \quad s \in \mathbb{C}, \quad \lambda \neq 0$

Sol. Fredholm Equation, Linear, Homogeneous, Singular, Second Kind.

d. $\int_0^x \frac{\sin(t)g(t)}{\sqrt{x-t}}dt = g(x)$

Sol. Volterra Equation, Linear, Homogeneous, Singular, Second Kind.

e. $\int_0^1 \left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{t}}\right) v(t)dt = \lambda f(x) + v(x)$

Sol. Fredholm Equation, Linear, Non-homogeneous, Singular, Second Kind.

Q.2 Consider the integral equation

$$v(x) = e^x + \lambda \int_1^2 \left(\frac{1+y}{x}\right) v(y)dy. \tag{1}$$

(a) Solve the integral equations (1) and identify its resolvent kernel.

Sol. Note that the integral equations (1) is Fredholm second kind linear with a separable kernel. Therefore, its solution is

$$v(x) = e^x + \frac{C\lambda}{x}, \tag{a}$$

where $C = \int_1^2 (1+y)v(y)dy$. In order to evaluate C, multiply (1) with $(1+x)$ and integrate over $(1,2)$, i.e.,

$$\begin{aligned}\int_1^2 (1+y)v(y)dy &= \int_1^2 (1+y)e^y dy + \lambda \int_1^2 \frac{1+x}{x} \int_1^2 (1+y)v(y)dydx \\ &= \int_1^2 (1+y)e^y dy + \lambda \int_1^2 \frac{1+x}{x}dx \int_1^2 (1+y)v(y)dy.\end{aligned}$$

On simplification, we get

$$\int_1^2 (1+y)v(y)dy \left[1 - \lambda \int_1^2 \frac{1+x}{x} dx\right] = \int_1^2 (1+y)e^y dy.$$

With the assumption that $\left[1 - \lambda \int_1^2 \frac{1+x}{x} dx\right] \neq 0$, we get

$$\int_1^2 (1+y)v(y)dy = \frac{1}{\left[1 - \lambda \int_1^2 \frac{1+x}{x} dx\right]} \int_1^2 (1+y)e^y dy.$$

Thus, the solution to the integral equation (1) is given by

$$\begin{aligned} v(x) =& e^x + \frac{\lambda}{x\left[1 - \lambda \int_1^2 \frac{1+x}{x} dx\right]} \int_1^2 (1+y)e^y dy \\ =& e^x + \int_1^2 \frac{\lambda(1+y)}{x\left[1 - \lambda \int_1^2 \frac{1+x}{x} dx\right]} e^y dy. \end{aligned} \tag{b}$$

From equation (b), it is apparent that the resolvent kernel associated with (1) is

$$R(x, y; \lambda) := \frac{\lambda(1+y)}{x\left[1 - \lambda \int_1^2 \frac{1+x}{x} dx\right]}, \tag{c}$$

and the solution to (1) can be written as

$$v(x) = e^x + \int_1^2 R(x, y; \lambda)e^y dy.$$

On evaluating the integrals involved in (b), one can easily get

$$\begin{aligned} v(x) =& e^x + \int_1^2 \frac{\lambda}{x\left[1 - \lambda(\ln|x| + x)\right]_1^2} (1+y)e^y dy \\ =& e^x + \frac{\lambda}{x(1 - \lambda(\ln|2| + 2 - 1))} \int_1^2 (1+y)e^y dy \\ =& e^x + \frac{\lambda}{x(1 - \lambda(\ln|2| + 1))} \left[ye^y\right]_1^2 \\ =& e^x + \frac{\lambda(2e^2 - e)}{x(1 - \lambda(\ln|2| + 1))}. \end{aligned} \tag{d}$$

(b) Find the characteristic values and associated non-trivial solutions (if any) of the associated homogeneous equation to (1).

Sol. The associated homogeneous equation to (1) is given by

$$v(x) = \lambda \int_1^2 \left(\frac{1+y}{x}\right) v(y)dy,$$

whose solution can be given by $v(x) = \dfrac{\lambda C}{x}$ where C is defined as in Part 2(a). In order find the value of C, we follow the same steps and get the equations

$$\int_1^2 (1+y)v(y)dy \left[1 - \lambda \int_1^2 \frac{1+x}{x} dx\right] = 0.$$

If $C = \displaystyle\int_1^2 (1+y)v(y)dy = 0$ then only trivial solution $v(x) = 0$ is possible. The non-trivial solutions are possible only if $\left[1 - \lambda \displaystyle\int_1^2 \frac{1+x}{x} dx\right] = 0$, i.e.,

$$\lambda = \left(\int_1^2 \frac{1+x}{x} dx\right)^{-1} = \frac{1}{(\ln 2 + 1)}.$$

This is the only characteristic value. The corresponding non-trivial solutions are

$$v(x) = \frac{C}{x(\ln 2 + 1)}, \qquad \text{for all } C \in \mathbb{R}. \tag{e}$$

Remark that these are infinite many solutions but there is only one linearly independent solution (say) $v(x) = \dfrac{1}{x(\ln 2 + 1)}$.

Q.3 Consider the integral equation

$$h(y) = \sin y + \lambda \int_0^\pi \cos y \sin z \, h(z)dz. \tag{2}$$

(a) Solve the integral equation and identify the resolvent kernel.

Sol. Note that (2) is also Fredholm linear second kind integral equation with a separable kernel. Therefore, the solution to (2) is given by

$$h(y) = \sin y + C\lambda \cos y \quad \text{with} \quad C := \int_0^\pi \sin z h(z)dz. \tag{f}$$

In order to find the value of C, multiply equation (2) with $\sin x$ and integrate over $[0, \pi]$. This renders

$$\begin{aligned} C = \int_0^\pi \sin z h(z)dz &= \int_0^\pi \sin^2 z dz + \lambda \int_0^\pi \sin y \cos y \int_0^\pi \sin z h(z) dz dy \\ &= \int_0^\pi \sin^2 z dz + \lambda \int_0^\pi \sin y \cos y \, dy \int_0^\pi \sin z h(z) dz \\ &= \int_0^\pi \sin^2 z dz + C\lambda \int_0^\pi \sin y \cos y \, dy. \end{aligned}$$

On simplification, we arrive at

$$C\left[1-\lambda\int_0^\pi \sin y\,\cos y\,dy\right]=\int_0^\pi \sin^2 z dz.$$

With an assumption that $\left[1-\lambda\int_0^\pi \sin y\,\cos y\,dy\right]\neq 0$, we get

$$C=\frac{1}{\left[1-\lambda\int_0^\pi \sin y\,\cos y\,dy\right]}\int_0^\pi \sin^2 z dz. \tag{g}$$

Therefore, the solution to (1) is given by

$$\begin{aligned} h(y) &= \sin y+\frac{\lambda\cos y}{\left[1-\lambda\int_0^\pi \sin y\,\cos y\,dy\right]}\int_0^\pi \sin^2 z dz \\ &= \sin y+\int_0^\pi \frac{\lambda\cos y\,\sin z}{\left[1-\lambda\int_0^\pi \sin y\,\cos y\,dy\right]}\sin z\,dz. \end{aligned} \tag{h}$$

Therefore, from Eq. (h), it is clear that the resolvent kernel of Eq. (2) is given by

$$R(y,z;\lambda):=\frac{\lambda\cos y\,\sin z}{\left[1-\lambda\int_0^\pi \sin y\,\cos y\,dy\right]}. \tag{i}$$

Moreover, on further simplification, one arrives at

$$\begin{aligned} h(y) &= \sin y+\frac{2\lambda\cos y}{2-\lambda\int_0^\pi \sin(2y)\,dy}\int_0^\pi \frac{1-\cos(2z)}{2}\,dz \\ &= \sin y+\frac{4\lambda\cos y}{4+\lambda\left[\cos(2y)\right]_0^\pi}\left[\frac{2z-\sin(2z)}{4}\right]_0^\pi \\ &= \sin y+\frac{\lambda\pi}{2}\cos y. \end{aligned}$$

(b) Find eigenvalues and the corresponding eigen-functions (if any).

Sol. In this case, the homogeneous equation associated to Eq. (2) does not have an eigenvalue and therefore, admits only a trivial solution. In fact, it is evident from Part 3(a) that $\left[1-\lambda\int_0^\pi \sin y\,\cos y\,dy\right]\neq 0$. Indeed,

$$1-\lambda\int_0^\pi \sin y\,\cos y\,dy=1-\frac{\lambda}{2}\int_0^\pi \sin(2y)\,dy=1+\frac{\lambda}{4}\left[\cos(2y)\right]_0^\pi=1-\frac{\lambda}{4}(0)=1\neq 0.$$

Therefore, following the procedure as in Q2(b), we will arrive at the situation

$$C\left[1-\lambda\int_0^{\pi}\sin y\,\cos y\,dy\right]=0$$

and that will lead only to $C=0$ for every choice of λ! Thus, there is only trivial solution to the homogeneous equation which does not have any eigenvalues and eigen-functions.

Q.4 Consider the problem of finding $\varphi(x)$ from the integral equation

$$\varphi(x)=f(x)-\lambda\int_0^x\varphi(y)dy, \tag{3}$$

where $f(x)$ is a known, real continuous function with continuous first derivative and $f(0)=0$.

(a) Show that this problem may be re-expressed as an ordinary differential equation with suitable boundary condition. (*Hint: Recall the Leibniz rule*

$$\frac{d}{dx}\int_{\alpha(x)}^{\beta(x)}\kappa(x,y)dy=\frac{d\beta}{dx}\kappa(x,\beta(x))-\frac{d\alpha}{dx}\kappa(x,\alpha(x))+\int_{\alpha(x)}^{\beta(x)}\frac{\partial}{\partial x}(\kappa(x,y))dy,$$

discussed in the class).

Sol. Differentiating Eq. (3) using the Leibniz rule, one gets

$$\begin{aligned}\varphi'(x)=&f'(x)-\lambda\left[\frac{d}{dx}(x)\varphi(x)-\frac{d}{dx}(0)\varphi(0)+\int_0^x\frac{d}{dx}(\varphi(y))dy\right]\\=&f'(x)-\lambda[\varphi(x)-0+0]\\=&f'(x)-\lambda\varphi(x).\end{aligned}$$

Moreover, since $f(0)=0$, the solution φ to the integral equation (3) satisfies the condition

$$\varphi(0)=f(0)-\lambda\int_0^0\varphi(y)dy=0-0=0.$$

Thus, the integral equation (3) can be re-expressed as a boundary value problem

$$\begin{cases}\varphi'(x)+\lambda\varphi(x)=f'(x),\\ \varphi(0)=0.\end{cases} \tag{j}$$

(b) Express the resulting differential equation as $L[\varphi]=f'$.

Sol. In view of the boundary value problem (j), we define the differential operator

$$L[\cdot]:=\frac{d}{dx}[\cdot]+\lambda I[\cdot], \tag{k}$$

where I is the identity map. Having defined L in Eq. (k), one can rewrite Eq. (j) as

$$L[\varphi](x)=f'(x).$$

(c) Show that the operator L is linear.

Sol. It is evident that L is a linear differential operator. Indeed, for all sufficiently smooth functions φ_1 and φ_2, and constants $c_1, c_2 \in \mathbb{R}$,

$$\begin{aligned} L[c_1\varphi_1 + c_2\varphi_2](x) =& \frac{d}{dx}\left[c_1\varphi_1 + c_2\varphi_2\right] + \lambda\left[c_1\varphi_1 + c_2\varphi_2\right] \\ =& c_1\frac{d}{dx}\left[\varphi_1\right] + c_2\frac{d}{dx}\left[\varphi_2\right] + c_1\lambda\left[\varphi_1\right] + c_2\lambda\left[\varphi_2\right] \\ =& c_1\left(\frac{d}{dx}\left[\varphi_1\right] + \lambda\varphi_1\right) + c_2\left(\frac{d}{dx}\left[\varphi_2\right] + \lambda\varphi_2\right) \\ =& c_1 L[\varphi_1] + c_1 L[\varphi_1], \end{aligned}$$

and

$$L[0] = \frac{d}{dx}(0) + \lambda 0 = 0.$$

"Your problem isn't the problem, it's your attitude about the problem." — Ann Brashares.