School of Natural Sciences
Department of Mathematics
Assignment 1 (Integral Equations), Spring 2020
Due Date: February 06, 2020

Give your best short while understanding and solving the problem set. I want your honest effort (no copying). It won't matter if you make mistakes, after all everybody does make mistakes when learning something new. That's how we learn!

Q.1 Classify the following integral equations as Volterra, Fredholm, linear, non-linear, homogeneous, non-homogeneous, singular, non-singular, first kind, and second kind.

a. $\int_a^x (x^2t - xt^2)y(t)dt = f(x)$

b. $\int_0^1 \frac{\sqrt{f(t)}}{x-t}dt = 1 - x + f(x)$

c. $\lambda \int_0^{+\infty} e^{-st} f(t)dt = f(s), \qquad \lambda \in \mathbb{R}, \quad s \in \mathbb{C}, \quad \lambda \neq 0$

d. $\int_0^x \frac{\sin(t)g(t)}{\sqrt{x-t}}dt = g(x)$

e. $\int_0^1 \left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{t}}\right) v(t)dt = \lambda f(x) + v(t)$

Q.2 Consider the integral equation

$$v(x) = e^x + \lambda \int_1^2 \left(\frac{1+y}{x}\right) v(y)dy. \tag{1}$$

(a) Solve the integral equations (1) and identify its resolvent kernel.

(b) Find the characteristic values and associated non-trivial solutions (if any) of the associated homogeneous equation to (1).

Q.3 Consider the integral equation

$$h(y) = \sin y + \lambda \int_0^\pi \cos y \sin z \, h(z)dz. \tag{2}$$

(a) Solve the integral equation and identify the resolvent kernel.

(b) Find eigenvalues and the corresponding eigen-functions (if any).

Q.4 Consider the problem of finding $\varphi(x)$ from the integral equations

$$\varphi(x) = f(x) - \lambda \int_0^x \varphi(y)dy, \tag{3}$$

where $f(x)$ is a known, real continuous function with continuous first derivative and $f(0) = 0$.

(a) Show that this problem may be re-expressed as an ordinary differential equation with suitable boundary condition. (*Hint: Recall the Leibniz rule*

$$\frac{d}{dx}\int_{\alpha(x)}^{\beta(x)} \kappa(x,y)dy = \frac{d\beta}{dx}\kappa(x,\beta(x)) - \frac{d\alpha}{dx}\kappa(x,\alpha(x)) + \int_{\alpha(x)}^{\beta(x)} \frac{\partial}{\partial x}(\kappa(x,y))dy,$$

discussed in the class).

(b) Express the resulting differential equation as $L[\varphi] = f'$.

(c) Show that the operator L is linear.

"Your problem isn't the problem, it's your attitude about the problem." — Ann Brashares.