School of Natural Sciences
Department of Mathematics
Assignment 2 (Integral Equations), Spring 2020
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Green's Function of Sturm-Liouville Problems

Introduction

The homogeneous differential equation
$$\frac{d^2y}{dx^2} = 0,$$
can be solved very easily and the solution is $y = Ax + B$ (a straight line). The constants can be found if boundary conditions are given. Similarly, the homogeneous equation

$$\frac{d^2y}{dx^2} + k^2y = 0, \tag{1}$$

can be solved to get

$$y = A\sin kx + B\cos kx. \tag{2}$$

Thus, there are simple techniques available to solve homogeneous equations. But, if we replace them with source terms like

$$\frac{d^2y}{dx^2} = \ln x, \quad \text{and} \quad \frac{d^2y}{dx^2} + k^2y = \tan x, \tag{3}$$

then the problems become difficult to solve.

The most general form of the second order linear ordinary differential operator is the *Sturm-Liouville (SL) operator* given by

$$\mathcal{L}[y[(x) := \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y. \tag{4}$$

Any second order ordinary linear differential equation

$$P_1(x)\frac{d^2y}{dx^2} + P_2(x)\frac{dy}{dx} + Q(x)y = F(x), \tag{5}$$

can be converted to an SL problem $\mathcal{L}[y](x) = f(x)$ using the integrating factor

$$\mu(x) := \frac{1}{P_1(x)}\exp\left(\int \frac{P_2(x)}{P_1(x)}dx\right), \tag{6}$$

with

$$p(x) := \mu(x)P_1(x), \qquad q(x) := \mu(x)Q(x) \quad \text{and} \quad f(x) := \mu(x)F(x). \tag{7}$$

Of course, $P_1(x)$ is assumed positive for all x. We restrict our discussion to the case when x belongs to a bounded interval $[a, b]$, and P_1, P_2, Q and F are assumed continuous.

Our goal is to determine a function $G(x, s)$ so that the general solution of

$$\mathcal{L}[y] = \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] - q(x)y = f(x), \qquad \forall x \in (a, b) \tag{8}$$

$$\alpha y(a) + \beta\frac{dy}{dx}(a) = 0, \tag{9}$$

$$\gamma y(b) + \delta\frac{dy}{dx}(b) = 0 \tag{10}$$

can be written as

$$y(x) = \int_a^b G(x, s)f(s)ds, \tag{11}$$

where α, β, γ and δ are known constants. Such a function is called a *Green's function*, named after the British mathematical physicist George Green (1793-1841). Green's function can be of great utility as it reduces the problem of solving (8) subject to boundary conditions (9)-(10) to the task of computing a single integral (11).

For simplicity and for understanding the procedural details to arrive at the Green's function, we consider a simple example of the operator $\left(\frac{d^2y}{dx^2} + k^2\right)$ (known as the one-dimensional Helmholtz operator, generally, linked to the motion of strings and waves, and $k = \omega/c$ is the wave-number defined in terms of the frequency of the mechanical oscillations ω and speed of the wave c).

Problem Set

We consider the boundary value problem

$$\frac{d^2y}{dx^2} + k^2y = f(x), \qquad \forall x \in (0, \pi), \tag{12}$$

$$y(0) = 0, \tag{13}$$

$$\frac{dy}{dx}(\pi) = 0, \tag{14}$$

where $k \neq 0$. (Note that $p(x) = 1$, $q(x) = -k^2$, $a = 0$, $b = \pi$, $\alpha = 1$, $\beta = 0$, $\gamma = 0$, $\delta = 1$.)

Q1. Find two linearly independent solutions of the associated homogeneous equation $\frac{d^2y}{dx^2} + k^2y = 0$ and use them for deriving the complementary solution y_c.

Sol. Note that (12) is a constant coefficient equation and the characteristic equation of the associated homogeneous equation is $m^2 + k^2 = 0$. Thus, the roots of the characteristic equation are given by $m = \pm\imath k$ and the complementary solution is given by

$$y_c(x) := c_1 \sin(kx) + c_2 \cos(kx).$$

The two linearly independent solutions are $\sin(kx)$ and $\cos(kx)$.

Q2. Use y_c to get two solutions y_1 and y_2 satisfying individual boundary conditions (13) and (14), respectively. (Hint: Impose boundary condition (13) on y_c and eliminate one constant to get y_1. Then, impose boundary condition (14) on y_c (afresh) and get y_2).

Sol. Imposing boundary conditions (13) on y_c provides $0 = y_c(0) = c_1 \sin(0) + c_2 \cos(0) = c_2$. Thus,

$$y_c := c_1 \sin(kx),$$

is a solution to the associated homogeneous equations that satisfies boundary condition (13). Therefore, we set $y_1 = \sin(kx)$.

Similarly, imposing boundary conditions (14) on y_c gives

$$0 = y_c'(\pi) = c_1 \cos(k\pi) - c_2 \sin(k\pi) \implies c_1 = c_2 \left(\frac{\sin(k\pi)}{\cos(k\pi)}\right),$$

provided $k \neq (2n+1)/2$ for all $n \in \mathbb{Z}$. Thus, substituting the value of c_1 back in y_c renders

$$\begin{aligned} y_c(x) =& c_2 \left(\frac{\sin(k\pi)}{\cos(k\pi)} \sin(kx) + \cos(kx)\right) \\ =& c_2 \frac{(\cos(kx)\cos(k\pi) + \sin(kx)\sin(k\pi))}{\cos(k\pi)} \\ =& \frac{c_2}{\cos(k\pi)} \cos(k(x-\pi)). \end{aligned}$$

Therefore, we choose $y_2 := \cos(k\pi)/\cos(k(x-\pi))$ It can be easily verified that y_1 satisfies (13) and y_2 satisfies (14) in addition to the homogeneous equation.

We could have eliminated c_2 instead of c_1. What do you think would change in y_2? We could also have neglected the constant $\cos(k\pi)$ in the denominator to simply choose $y_2(x) = \cos(k(x-\pi))$, what effect it will have in your opinion?

Q3. Find the Wronskian, $w(y_1, y_2)$, of the solutions y_1 and y_2 obtained in Q2. Show that y_1 and y_2 are linearly independent.

Sol. Remember that $w(y_1, y_2) = y_1 y_2' - y_1' y_2$. Since

$$y_1(x) = \sin(kx), \quad y_2(x) = \frac{\cos(k(x-\pi))}{\cos(k\pi)}, \quad y_1'(x) = k\cos(kx) \text{ and } y_2'(x) = -\frac{k\sin(k(x-\pi))}{\cos(k\pi)},$$

we have

$$\begin{aligned} w(y_1, y_2) =& -k \left[\frac{\sin(kx)\sin(k(x-\pi)) + \cos(kx)\cos(k(x-\pi))}{\cos(k\pi)}\right] \\ =& -k \left[\frac{\cos(kx - k(x-\pi))}{\cos(k\pi)}\right] = -k. \end{aligned}$$

Since $w(y_1, y_2) \neq 0$ (recall that $k \neq 0$), the functions y_1 and y_2 are linearly independent.

Q4. In order to derive a particular solution of (12), define $y_p := c_1(x)y_1(x) + c_2(x)y_2(x)$ as in the method of variation of parameters for finding particular solutions. Show that

$$c_1(x) = -\int_0^x \frac{y_2(s)f(s)}{w(y_1,y_2)(s)}ds \quad \text{and} \quad c_2(x) = \int_0^x \frac{y_1(s)f(s)}{w(y_1,y_2)(s)}ds. \tag{15}$$

Sol. Let us recall the method of variation of parameters. We first differentiate $y_p(x) := c_1(x)y_1(x) + c_2(x)y_2(x)$ to get

$$y_p'(x) = \Big(c_1(x)y_1'(x) + c_2(x)y_2'(x)\Big) + \Big(c_1'(x)y_1(x) + c_2'(x)y_2(x)\Big).$$

and set

$$c_1'(x)y_1(x) + c_2'(x)y_2(x) = 0. \tag{a}$$

Note that this is an assumption to simplify the derivative $y_p'(x)$, (however, it does not pose problem!). We differentiate the reduced form of y_p' again to get

$$y_p''(x) = \Big(c_1(x)y_1''(x) + c_2(x)y_2''(x)\Big) + \Big(c_1'(x)y_1'(x) + c_2'(x)y_2'(x)\Big). \tag{16}$$

Substituting the expressions for y_p and y_p'' in (12) and simplifying the resultant, we arrive at

$$c_1(x)\underbrace{\left(\frac{d^2y_1}{dx^2} + k^2y_1\right)}_{=0} + c_2(x)\underbrace{\left(\frac{d^2y_2}{dx^2} + k^2y_2\right)}_{=0} + \Big(c_1'(x)y_1'(x) + c_2'(x)y_2'(x)\Big) = f(x).$$

This gives

$$c_1'(x)y_1'(x) + c_2'(x)y_2'(x) = f(x). \tag{b}$$

We find the functions c_1' and c_2' by solving the system of equations (a) and (b), which has a unique solution because $w(y_1,y_2) \neq 0$. By Cramer's rule

$$c_1'(x) = -\frac{y_2(x)f(x)}{w(y_1,y_2)(x)} \quad \text{and} \quad c_2'(x) = \frac{y_1(x)f(x)}{w(y_1,y_2)(x)},$$

from where we get the required forms upon integration from 0 to x. Therefore, y_p appears to be

$$\begin{aligned} y_p(x) &= -\,y_1(x)\int_0^x \frac{y_2(s)f(s)}{w(y_1,y_2)(s)}ds + y_2(x)\int_0^x \frac{y_1(s)f(s)}{w(y_1,y_2)(s)}ds \\ &= -\int_0^x \frac{\sin(kx)\cos(k(s-\pi))}{k\cos(k\pi)}f(s)ds + \int_0^x \frac{\sin(ks)\cos(k(x-\pi))}{k\cos(k\pi)}f(s)ds. \end{aligned}$$

Q5. Write down the general solution of the equations (12) as $y(x) = Ay_1(x) + By_2(x) + y_p(x)$. Impose the boundary conditions (13)-(14) simultaneously on $y(x)$ and find the values of constants A and B (perhaps in terms of integrals).

Sol. In order to find the values of A and B so that $y(x) = Ay_1(x) + By_2(x) + y_p(x)$ satisfy the boundary conditions (13)-(14), we must have $y(0) = 0 = v'(\pi)$. Note that $y_1(0) = 0$, $y_2(0) = 1$ and $y_p(0) = 0$. Therefore, $y(0) = 0$ implies

$$Ay_1(0) + By_2(0) + y_p(0) = 0 \implies A(0) + B(1) + (0) = 0 \implies B = 0.$$

Also note that $y_1'(\pi) = k\cos(k\pi)$ and

$$y_p'(\pi) = c_1(\pi)y_1'(\pi) + 0 = -\int_0^\pi \cos(k(s-\pi)f(s)ds.$$

Therefore, $y'(\pi) = 0$ provides

$$0 = Ay_1'(\pi) + y_p'(\pi) = Ak\cos(k\pi) - \int_0^\pi \cos(k(s-\pi)f(s)ds \implies A = \int_0^\pi \frac{\cos(k(s-\pi))}{k\cos(k\pi)} f(s)ds.$$

Hence, the general solution to (12) is given by

$$\begin{aligned} y(x) =& Ay_1(x) + y_p(x) \\ =& \int_0^\pi \frac{\sin(kx)\cos(k(s-\pi))}{k\cos(k\pi)} f(s)ds + \\ & - \int_0^x \frac{\sin(kx)\cos(k(s-\pi))}{k\cos(k\pi)} f(s)ds + \int_0^x \frac{\sin(ks)\cos(k(x-\pi))}{k\cos(k\pi)} f(s)ds \\ =& \int_x^\pi \frac{\sin(kx)\cos(k(s-\pi))}{k\cos(k\pi)} f(s)ds + \int_0^x \frac{\sin(ks)\cos(k(x-\pi))}{k\cos(k\pi)} f(s)ds. \end{aligned}$$

Q6. Show that $y(x)$ calculated in Q5 can be expressed in the form

$$y(x) := \int_0^\pi G(x,s)f(s)ds, \tag{17}$$

where $G(x,s)$ can be written in the form

$$G(x,s) := \begin{cases} g_1(x,s), & s < x, \\ g_2(x,s), & x < s. \end{cases} \tag{18}$$

Sol. The general solution obtained in the previous question can be rearranged as

$$y(x) = \int_0^\pi G(x,s)f(s)ds.$$

by defining the piece-wise function

$$G(x,s) = \begin{cases} \dfrac{y_1(s)y_2(x)}{w(y_1,y_2)(s)} & s < x, \\ \\ \dfrac{y_1(s)y_2(x)}{w(y_1,y_2)(s)} & x < s, \end{cases} \quad \text{or} \quad G(x,s) = \begin{cases} \dfrac{\sin(ks)\cos(k(x-\pi))}{k\cos(k\pi)} & s < x, \\ \\ \dfrac{\sin(kx)\cos(k(s-\pi))}{k\cos(k\pi)} & x < s. \end{cases}$$

Q7. (Optional) Show that $G(x, s)$ is symmetric, i.e., $G(x, s) = G(s, x)$.

Sol. Note that

$$G(s,x) = \begin{cases} \dfrac{\sin(kx)\cos(k(s-\pi))}{k\cos(k\pi)} & x < s, \\ \\ \dfrac{\sin(ks)\cos(k(x-\pi))}{k\cos(k\pi)} & s < x, \end{cases}$$

which is essentially $G(x, s)$.

Q8. (Optional) Show that $G(x, s)$ is continuous at $x = s$.

Sol. Note that

$$G(x,x) = \frac{\sin(kx)\cos(k(x-\pi))}{k\cos(k\pi)}.$$

Moreover, for $\epsilon > 0$

$$\lim_{x\to s+\epsilon} G(x,s) = \lim_{\epsilon\to 0} G(s+\epsilon, s) = \lim_{\epsilon\to 0} \frac{\sin(k(s)\cos(k(s+\epsilon-\pi))}{k\cos(k\pi)} = \frac{\sin(k(s)\cos(k(s-\pi))}{k\cos(k\pi)},$$
$$\lim_{x\to s-\epsilon} G(x,s) = \lim_{\epsilon\to 0} G(s-\epsilon, s) = \lim_{\epsilon\to 0} \frac{\sin(k(s-\epsilon)\cos(k(s-\pi))}{k\cos(k\pi)} = \frac{\sin(k(s)\cos(k(s-\pi))}{k\cos(k\pi)}.$$

Therefore, $G(x, s)$ is continuous at $x = s$.

Q9. (Optional) Show that $\left.\frac{dg_1}{dx}(x,s)\right|_{x=s} = \left.\frac{dg_2}{dx}(x,s)\right|_{x=s} + 1$ since $p(x) = 1$ here. (Hint: Integrate the equation $\frac{d^2}{dx^2}[G](x,s) + k^2 G(x,s) = \delta(x-s)$ over infinitesimally small interval $[s-\epsilon, s+\epsilon]$ and take limit $\epsilon \to 0$. Here, δ is the Dirac mass.)

Sol. Note that

$$\lim_{\epsilon\to 0} \int_{s-\epsilon}^{s+\epsilon} \left(\frac{d^2}{dx^2}[G](x,s) + k^2 G(x,s) \right) dx = \lim_{\epsilon\to 0} \int_{s-\epsilon}^{s+\epsilon} \delta(x-s)dx.$$

Since,

$$\int_{s-\epsilon}^{s+\epsilon} \delta(x-s)dx = 1,$$

we have

$$\lim_{\epsilon\to 0} \left[\frac{d}{dx}[G](x,s) \right]_{s-\epsilon}^{s+\epsilon} + k^2 \lim_{\epsilon\to 0} \left(\int_{s-\epsilon}^{s+\epsilon} G(x,s) \right) dx = \lim_{\epsilon\to 0} 1.$$

Since, $G(x,s)$ is continuous at $x = s$, it has a continuous primitive (say) $K(x,s)$, i.e., we have

$$\frac{dK(x,s)}{dx} = G(x,s).$$

Therefore,

$$\lim_{\epsilon\to 0}\left(\int_{s-\epsilon}^{s+\epsilon} G(x,s)\right)dx = \lim_{\epsilon\to 0}\left(\int_{s-\epsilon}^{s+\epsilon} \frac{dK(x,s)}{dx}\right)dx = \lim_{\epsilon\to 0}\Big[K(x,s)\Big]_{s-\epsilon}^{s+\epsilon}.$$

Since, $K(x,s)$ is continuous at $x = s$, we have

$$\lim_{\epsilon\to 0}\Big[K(x,s)\Big]_{s-\epsilon}^{s+\epsilon} = \lim_{\epsilon\to 0} K(x,s)\Big|_{x=s+\epsilon} - \lim_{\epsilon\to 0} K(x,s)\Big|_{x=s-\epsilon} = K(s,s) - K(s,s) = 0$$

Therefore,

$$\lim_{\epsilon\to 0}\left[\frac{d}{dx}[G](x,s)\right]_{s-\epsilon}^{s+\epsilon} + k^2(0) = 1,$$

or,

$$\lim_{\epsilon\to 0}\left[\frac{dG}{dx}(x,s)\Big|_{x=s+\epsilon} - \frac{dG}{dx}(x,s)\Big|_{x=s-\epsilon}\right] = 1.$$

Setting $g_1(x,s) := G(x,s)$ for $s < x$ and $g_2(x,s) := G(x,s)$ for $x < s$, we can write the above equation as

$$\lim_{\epsilon\to 0}\left[\frac{dg_1}{dx}(x,s)\Big|_{x=s+\epsilon} - \frac{dg_2}{dx}(x,s)\Big|_{x=s-\epsilon}\right] = 1.$$

Since, g_1 and g_2 are continuous over the intervals $[0,s]$ and $[s,\pi]$, we finally can pass on the limit to arrive at

$$\left[\frac{dg_1}{dx}(x,s)\Big|_{x=s} - \frac{dg_2}{dx}(x,s)\Big|_{x=s}\right] = 1.$$

"If you really want to do something, you'll find a way. If you don't, you'll find an excuse." — Jim Rohn.