School of Natural Sciences
Department of Mathematics
Assignment 2 (Integral Equations), Spring 2020
Due Date: February 24, 2020

Green's Function of Sturm-Liouville Problems

Introduction

The homogeneous differential equation

$$\frac{d^2y}{dx^2} = 0,$$

can be solved very easily and the solution is $y = Ax + B$ (a straight line). The constants can be found if boundary conditions are given. Similarly, the homogeneous equation

$$\frac{d^2y}{dx^2} + k^2 y = 0, \tag{1}$$

can be solved to get

$$y = A \sin kx + B \cos kx. \tag{2}$$

Thus, there are simple techniques available to solve homogeneous equations. But, if we replace them with source terms like

$$\frac{d^2y}{dx^2} = \ln x, \quad \text{and} \quad \frac{d^2y}{dx^2} + k^2 y = \tan x, \tag{3}$$

then the problems become difficult to solve.

The most general form of the second order linear ordinary differential operator is the *Sturm-Liouville (SL) operator* given by

$$\mathcal{L}[y[(x) := \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y. \tag{4}$$

Any second order ordinary linear differential equation

$$P_1(x)\frac{d^2y}{dx^2} + P_2(x)\frac{dy}{dx} + Q(x)y = F(x), \tag{5}$$

can be converted to an SL problem $\mathcal{L}[y](x) = f(x)$ using the integrating factor

$$\mu(x) := \frac{1}{P_1(x)} \exp\left(\int \frac{P_2(x)}{P_1(x)} dx\right), \tag{6}$$

with

$$p(x) := \mu(x)P_1(x), \qquad q(x) := \mu(x)Q(x) \quad \text{and} \quad f(x) := \mu(x)F(x). \tag{7}$$

Of course, $P_1(x)$ is assumed positive for all x. We restrict our discussion to the case when x belongs to a bounded interval $[a, b]$, and P_1, P_2, Q and F are assumed continuous.

Our goal is to determine a function $G(x, s)$ so that the general solution of

$$\mathcal{L}[y] = \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] - q(x)y = f(x), \qquad \forall x \in (a, b) \tag{8}$$

$$\alpha y(a) + \beta \frac{dy}{dx}(a) = 0, \tag{9}$$

$$\gamma y(b) + \delta \frac{dy}{dx}(b) = 0 \tag{10}$$

can be written as

$$y(x) = \int_a^b G(x, s) f(s) ds, \tag{11}$$

where α, β, γ and δ are known constants. Such a function is called a *Green's function*, named after the British mathematical physicist George Green (1793-1841). Green's function can be of great utility as it reduces the problem of solving (8) subject to boundary conditions (9)-(10) to the task of computing a single integral (11).

For simplicity and for understanding the procedural details to arrive at the Green's function, we consider a simple example of the operator $\left(\frac{d^2y}{dx^2} + k^2\right)$ (known as the one-dimensional Helmholtz operator, generally, linked to the motion of strings and waves, and $k = \omega/c$ is the wave-number defined in terms of the frequency of the mechanical oscillations ω and speed of the wave c).

Problem Set

We consider the boundary value problem

$$\frac{d^2y}{dx^2} + k^2 y = f(x), \qquad \forall x \in (0, \pi), \tag{12}$$

$$y(0) = 0, \tag{13}$$

$$\frac{dy}{dx}(\pi) = 0, \tag{14}$$

where $k \neq 0$. (Note that $p(x) = 1$, $q(x) = -k^2$, $a = 0$, $b = \pi$, $\alpha = 1$, $\beta = 0$, $\gamma = 0$, $\delta = 1$.)

Q1. Find two linearly independent solutions of the associated homogeneous equation $\frac{d^2y}{dx^2} + k^2 y = 0$ and use them for deriving the complementary solution y_c.

Q2. Use y_c to get two solutions y_1 and y_2 satisfying individual boundary conditions (13) and (14), respectively. (Hint: Impose boundary condition (13) on y_c and eliminate one constant to get y_1. Then, impose boundary condition (14) on y_c (afresh) and get y_2).

Q3. Find the Wronskian, $w(y_1, y_2)$, of the solutions y_1 and y_2 obtained in Q2. Show that y_1 and y_2 are linearly independent.

Q4. In order to derive a particular solution of (12), define $y_p := c_1(x)y_1(x) + c_2(x)y_2(x)$ as in the method of variation of parameters for finding particular solutions. Show that

$$c_1(x) = -\int_0^x \frac{y_2(s)f(s)}{w(y_1, y_2)(s)}ds \quad \text{and} \quad c_2(x) = \int_0^x \frac{y_1(s)f(s)}{w(y_1, y_2)(s)}ds. \tag{15}$$

Q5. Write down the general solution of the equations (12) as $y(x) = Ay_1(x) + By_2(x) + y_p(x)$. Impose the boundary conditions (13)-(14) simultaneously on $y(x)$ and find the values of constants A and B (perhaps in terms of integrals).

Q6. Show that $y(x)$ calculated in Q5 can be expressed in the form

$$y(x) := \int_0^\pi G(x, s)f(s)ds, \tag{16}$$

where $G(x, s)$ can be written in the form

$$G(x, s) := \begin{cases} g_1(x, s), & s < x, \\ g_2(x, s), & x < s. \end{cases} \tag{17}$$

Q7. (Optional) Show that $G(x, s)$ is symmetric, i.e., $G(x, s) = G(s, x)$.

Q8. (Optional) Show that $G(x, s)$ is continuous at $x = s$.

Q9. (Optional) Show that $\frac{dg_1}{dx}(x, s)\Big|_{x=s} = \frac{dg_2}{dx}(x, s)\Big|_{x=s} + 1$ since $p(x) = 1$ here. (Hint: Integrate the equation $\frac{d^2}{dx^2}[G](x, s) + k^2 G(x, s) = \delta(x - s)$ over infinitesimally small interval $[s - \epsilon, s + \epsilon]$ and take limit $\epsilon \to 0$. Here, δ is the Dirac mass.)

"If you really want to do something, you'll find a way. If you don't, you'll find an excuse." — Jim Rohn.