National University of Technology, Islamabad
Assignment I (Linear Algebra and ODE), Fall 2019
Solution-Key

Q.1 Give brief answers to the following questions.

(a) Give an example of two row equivalent matrices in echelon form. Note that although the reduced echelon form is unique, there will be several different row equivalent matrices in echelon form.

Ans. For example, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. There are infinitely many solutions to this problem; the particular one shown here was obtained by starting with a matrix (in this case the identity matrix, which happens to be in reduced row echelon form) and adding its third row to its second row. Since this is an elementary row operation they remain row equivalent, and since we haven't introduced any nonzero elements under the pivot elements it remains in row echelon form.

(b) Suppose a 4×6 coefficient matrix has 4 pivot columns. Is the corresponding system of equations consistent? Justify your answer.

Ans. The system is consistent with infinitely many solutions since there is a pivot in every row but not every column. The columns without pivot elements correspond to free variables.

(c) If a 7×5 augmented matrix has a pivot in every column, what can you say about the solution to the corresponding system of equations? Justify your answer.

Ans. Because the matrix at hand is augmented matrix and has a pivot in every column, the rightmost column in particular must be a pivot column. This means that the system is not consistent, since it means that the row corresponding to the last pivot element looks like

$$0 \quad 0 \quad 0 \quad 0 \mid b$$

where $b \neq 0$, which is absurd. Hence the system is inconsistent.

(d) If a consistent system of equations has more unknowns than equations, what can be said about the number of solutions?

Ans. A system with more unknowns than equations has either no solutions or infinitely many solutions. Since our system is specified to be consistent (meaning it has at least one solution), it must have infinite solutions then.

(e) Explain the concept of elementary matrices? Give three examples of elementary matrices obtained by different elementary row operations.

Ans. Matrix obtained by a single elementary row operation performed on an identity matrix is called an elementary matrix, that is, any matrix that can be made row-equivalent to an identity matrix by just a single elementary row operation is called elementary matrix. Since there are three basic and independent elementary row operations, we give examples of three different elementary matrices as

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{(rows 2 and 3 are swaped)}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{(row 3 is added in row 2)}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -7 \end{pmatrix} \quad \text{(row 4 is multiplied with a constant).}$$

Q.2 Find the values of k for which the system of equations $\begin{cases} x + ky = 1 \\ kx + y = 1 \end{cases}$ has no solution, exactly one solution and infinitely many solutions.

Ans. Note that the augmented matrix corresponding to the system is $[A|b] := \begin{pmatrix} 1 & k & 1 \\ k & 1 & 1 \end{pmatrix}$, that has the echelon form

$$[A|b] \sim \begin{pmatrix} 1 & k & 1 \\ 0 & 1-k^2 & 1-k \end{pmatrix} \qquad (R_2 \to R_2 - kR_1).$$

The system is inconsistent if $\text{Rank}(A) < \text{Rank}([A|b])$, i.e., $1 - k^2 = 0$ but $1 - k \neq 0$. This is only possible if $k = -1$. Hence, for $k = -1$ the system does not possess any solution. The system is consistent if $\text{Rank}(A) = \text{Rank}([A|b])$, i.e., either both $1 - k^2$ and $1 - k$ are non-zero or they are both zero. Note that when $1 - k^2 = 0 = 1 - k$, i.e., when $k = 1$, the coefficient matrix as well as the augmented matrix, both are rank deficient, i.e., there are two unknowns but one independent equation, i.e., there is one free variable. So, in that case there are infinite many solutions. If $1 - k^2 \neq 0$ and $1 - k \neq 0$, i.e. $k \neq \pm 1$ then $\text{Rank}(A) = 2 = \text{Rank}([A|b])$. Therefore, the system is consistent with no free variables. Thus, there is a unique solution.

Conclusion: There is no solution when $k = -1$, there is unique solution when $k \neq \pm 1$ and there are infinite many solutions when $k = 1$.

Q.3 Consider the following homogeneous system of linear equations (where a and b are non-zero constants)

$$\begin{aligned} x + 2y \qquad &= 0 \\ ax + 8y + 3z &= 0 \\ by + 5z &= 0. \end{aligned}$$

(a) Find a value for a which will make it necessary during Gaussian elimination to interchange rows in the coefficient matrix.

Ans. The coefficient matrix of the system is $A := \begin{pmatrix} 1 & 2 & 0 \\ a & 8 & 3 \\ 0 & b & 5 \end{pmatrix}$. In order to reduce it to echelon form, we proceed as follows:

$$A := \begin{pmatrix} 1 & 2 & 0 \\ a & 8 & 3 \\ 0 & b & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & 8-2a & 3 \\ 0 & b & 5 \end{pmatrix} \qquad (R_2 \to R_2 - aR_1).$$

Note that, if $8 - 2a = 0$, i.e., $a = 4$, it would be necessary (since b is non-zero) to interchange row two and row three.

(b) Suppose that a is different than the value you found in part (a). Find a value for b (of course in terms of a) so that the system has a non-trivial solution.

Ans. Suppose $a \neq 4$. We proceed with the Gaussian elimination as

$$A \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & \dfrac{3}{8-2a} \\ 0 & b & 5 \end{pmatrix} \qquad (R_2 \to \frac{1}{8-2a}R_2)$$

$$\sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & \dfrac{3}{8-2a} \\ 0 & 0 & 5 - \dfrac{3b}{8-2a} \end{pmatrix} \qquad (R_3 \to R_3 - bR_2).$$

Note that the coefficient matrix A has rank-deficiency if $5 - \dfrac{3b}{8-2a} = 0$. In that case, the given homogeneous system of linear equations will have one free variable and, therefore, there will be infinite many solutions. Hence, there are non-trivial solutions to the system if $b = \frac{40}{3} - \frac{10a}{3}$.

(c) Suppose that a is different than the value you found in part (a) and that $b = 100$. Suppose further that a is chosen so that the solution to the system is **not** unique. For what values of α and β, the general solution to the system (in terms of the free variable z) is $\left(\frac{z}{\alpha}, \frac{z}{\beta}, z\right)$?

Ans. Suppose $a \neq 4$ and $b = 100$. Then, for the system to have non-unique solutions, we must choose a such that $5 - \dfrac{300}{8-2a} = 0$. This renders $a = -26$ and so

$$A \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & \dfrac{1}{20} \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -\dfrac{1}{10} \\ 0 & 1 & \dfrac{1}{20} \\ 0 & 0 & 0 \end{pmatrix} \qquad (R_1 \to R_1 - 2R_2).$$

Therefore, we have $x = z/10$, $y = -z/20$ and $z =$ free variable. Thus, $\alpha = 10$ and $\beta = -20$.

(d) Write down the solution set obtained in the part (c) as a **span** of a set of vectors.

Ans. Let $z = 20r$ for some arbitrary $r \in \mathbb{R}$. Then, the solution of our system obtained in part (c) is given by $x = 2r$, $y = -r$, and $z = 20r$. Therefore, the solution set can be written as

$$\text{Solution Set} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{bmatrix} x \\ y \\ z \end{bmatrix} = r \begin{bmatrix} 2 \\ -1 \\ 20 \end{bmatrix}, \quad \forall r \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 2 \\ -1 \\ 20 \end{bmatrix} \right\}.$$

(e) Give a geometric interpretation of the solution set obtained in the part (c).

Ans. In parametric form, any solution vector $\mathbf{u} = \begin{bmatrix} x & y & z \end{bmatrix}^T \in \mathbb{R}^3$ is obtained as $\mathbf{u} = r\mathbf{v}$, where $\mathbf{v} = \begin{bmatrix} 2 & -1 & 20 \end{bmatrix}^T$. It is evident that span $\left\{ \begin{bmatrix} 2 & -1 & 20 \end{bmatrix}^T \right\}$ is one dimensional and $\mathbf{u} = r\mathbf{v}$ is a vector equation of the straight line parallel to vector $\mathbf{v}$ passing through the origin.

Q.4 Let $\mathbf{w} = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^T$, $\mathbf{x} = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}^T$, $\mathbf{y} = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}^T$, and $\mathbf{z} = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}^T$.

(a) We can show that $\{\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is not a spanning set for $\mathbb{R}^4$ by finding a vector $\mathbf{u}$ in $\mathbb{R}^4$ such that $\mathbf{u} \notin \text{span}\{\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}\}$. Find unknown a such that the vector $\mathbf{u} = \begin{bmatrix} 1 & 2 & 3 & a \end{bmatrix}^T$ is not in $\text{span}\{\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}\}$.

Ans If $\mathbf{u} \in \text{span}\{\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}\}$ then it can be written as a linear combination of the vectors $\mathbf{w}, \mathbf{x}, \mathbf{y}$, and $\mathbf{z}$ as

$$c_1\mathbf{w} + c_2\mathbf{x} + c_3\mathbf{y} + c_4\mathbf{z} = \mathbf{u}. \tag{1}$$

We find the value of a so that the system (1) is inconsistent, i.e., there does not exist constants c_1, c_2, c_3 and c_4 for $\mathbf{u}$ to be expressed as a linear combination (1).
In the matrix notation, system (1) can be written as

$$\begin{pmatrix} \mathbf{w} & \mathbf{x} & \mathbf{y} & \mathbf{z} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ a \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ a \end{pmatrix}.$$

Thus, the augmented matrix is given by

$$[A|\mathbf{u}] := \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 & a \end{pmatrix}.$$

We reduce it to reduced row echelon form as follows:

$$[A|\mathbf{u}] \sim \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 & a \end{pmatrix} \quad (R_2 \to R_2 - R_1)$$

$$\sim \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 & a \end{pmatrix} \quad (R_2 \to (-1)R_2)$$

$$\sim \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 1 & a \end{pmatrix} \quad (R_3 \to R_3 - R_2)$$

$$\sim \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 & a-4 \end{pmatrix} \quad (R_4 \to R_4 - R_3).$$

Note that the augmented matrix is in echelon form and there is a pivot element (if $a - 4 \neq 0$) in the last column. Therefore, the system is inconsistent if $a \neq 4$. Hence, every vector $\mathbf{u} = \begin{bmatrix} 1 & 2 & 3 & a \end{bmatrix}^T$ is not in $\text{span}\{\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}\}$ if $a \neq 4$.

(b) Show that $\{\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is a linearly dependent set by writing $\mathbf{z}$ as a linear combination of vectors $\mathbf{w}$, $\mathbf{x}$, and $\mathbf{y}$.

Ans. We write $\mathbf{z}$ as a linear combination of the vectors $\mathbf{w}, \mathbf{x}, \mathbf{y}$ as

$$c_1\mathbf{w} + c_2\mathbf{x} + c_3\mathbf{y} = \mathbf{z} \tag{2}$$

and find the values of c_1, c_2, c_3. In the matrix notation, system (2) can be written as

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

Thus, the augmented matrix is given by

$$[A|\mathbf{z}] := \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

We reduce it to reduced row echelon form as follows:

$$[A|\mathbf{z}] \sim \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad (R_2 \to R_2 - R_1)$$

$$\sim \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad (R_2 \to (-1)R_2)$$

$$\sim \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad (R_3 \to R_3 - R_2)$$

$$\sim \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (R_4 \to R_4 - R_3)$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (R_1 \to R_1 - R_2).$$

The reduced echelon form of the augmented matrix indicates that $c_1 = 1$, $c_2 = -1$, and $c_3 = 1$. Thus, $\mathbf{z} = \mathbf{w} - \mathbf{x} + \mathbf{y}$. Therefore, the set $\{\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly dependent.

"Your problem isn't the problem, it's your attitude about the problem." — Ann Brashares.