National University of Technology, Islamabad
Assignment II (Linear Algebra and ODE), Fall 2019
Solution-Key

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Q.1 Consider the matrix $\begin{bmatrix}1 & 3 & 2\\ a & 6 & 2\\ 0 & 9 & 5\end{bmatrix}$ where $a$ is a real number. For what value of $a$ is the matrix singular?

Ans. There can be different ways to determine $a$ so that the given matrix is singular. The first way is to find the determinant of the matrix and set it to be equal to zero so that the matrix is singular. By solving the resulting equation we can find the value of $a$. The second way is to consider the augmented matrix $[A \,|\, I_3]$ where $A$ denotes the given matrix and $I_3$ is the $3 \times 3$ identity matrix. We derive the reduced echelon form of this augmented matrix and suggest the value of $a$ so that the augmented matrix does not have 3 pivot positions. The third way is to simply find the echelon form of the given matrix $A$ and set the product of the diagonal elements of the echelon form equal to zero so that $A$ does not become row equivalent to $I_3$ and is, therefore, singular. Here, we try out the last technique. Consider

$$A := \begin{bmatrix}1 & 3 & 2\\ a & 6 & 2\\ 0 & 9 & 5\end{bmatrix} \sim \begin{bmatrix}1 & 3 & 2\\ 0 & 6-3a & 2-2a\\ 0 & 9 & 5\end{bmatrix} \qquad (R_2 \to R_2 - aR_1)$$

$$\sim \begin{bmatrix}1 & 3 & 2\\ 0 & 9 & 5\\ 0 & 6-3a & 2-2a\end{bmatrix} \qquad (R_{12})$$

$$\sim \begin{bmatrix}1 & 3 & 2\\ 0 & 9 & 5\\ 0 & 0 & 2(1-a) - \frac{5(2-a)}{3}\end{bmatrix} \qquad (R_3 \to R_3 - \frac{(2-a)}{3}R_2).$$

So, the matrix is singular if and only if $2(1-a) - \frac{5(2-a)}{3} = 0$, i.e., $6(1-a) - 5(2-a) = 0$ or equivalently, $a = -4$.

Q.2 Find the inverse of $\begin{bmatrix}1 & 0 & 0 & 0\\ \frac{1}{4} & 1 & 0 & 0\\ \frac{1}{3} & \frac{1}{3} & 1 & 0\\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1\end{bmatrix}$.

Ans. Let $H$ be the given matrix. Note that the matrix is lower triangular and has non-zero diagonal elements. Therefore, $H$ is invertible. In order to find its inverse, we can use the classical technique of minors and cofactors and then evaluate the inverse using $H^{-1} = (\det(H))^{-1}\,\mathrm{adj}(H)$. However, it will be very tedious. The second and more elegant way is to form the augmented matrix $[H \,|\, I_4]$ and reduce it to a reduced echelon form so that the inverse can be identified on the right half of the reduced augmented matrix. Albeit, it is better than the previous approach, it still does not exploit the structure of the matrix $H$.

We need to make use of the lower triangular form of the matrix $H$. Note that $H^T$ is an upper triangular matrix (thus in echelon form). Moreover, $(H^T)^{-1} = (H^{-1})^T$. Therefore, instead of calculating the inverse of $H$, we find the inverse of $H^T$. This way we can reduce our work done by half as compared to directly inverting $H$ using reduced echelon form of the augmented matrix $[H \,|\, I_4]$. Towards this end, consider the augmented matrix $[H^T \,|\, I_4]$ as

$$
\begin{aligned}
[H^T \,|\, I_4] &= \begin{bmatrix} 1 & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{2} & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 0 & \frac{1}{4} & \frac{3}{8} & 1 & -\frac{1}{4} & 0 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{2} & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} && (R_1 - \frac{1}{4}R_2) \\
&\sim \begin{bmatrix} 1 & 0 & 0 & \frac{1}{4} & 1 & -\frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & 1 & 0 & \frac{1}{3} & 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} && (R_1 - \frac{1}{4}R_3 \text{ and } R_2 - \frac{1}{3}R_3) \\
&\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & 0 & 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} && (R_1 - \frac{1}{4}R_4,\ R_2 - \frac{1}{3}R_4, \text{ and } R_3 - \frac{1}{2}R_4).
\end{aligned}
$$

Therefore, we have

$$
\left(H^T\right)^{-1} = \begin{bmatrix} 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \left(H^{-1}\right)^T.
$$

Taking the transpose on both sides we arrive at

$$
H^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{4} & 1 & 0 & 0 \\ -\frac{1}{4} & -\frac{1}{3} & 1 & 0 \\ -\frac{1}{4} & -\frac{1}{3} & -\frac{1}{2} & 1 \end{bmatrix}.
$$

Q.3 Recall that a set $\mathcal{B}$ of vectors of a subspace $\mathcal{S}$ of $\mathbb{R}^n$ is called a *basis* of $\mathcal{S}$ if $\mathcal{B}$ is linearly independent and it spans $\mathcal{S}$. Let $\mathbf{u} = \begin{bmatrix} 2 & 0 & -1 \end{bmatrix}^T$, $\mathbf{v} = \begin{bmatrix} 3 & 1 & 0 \end{bmatrix}^T$, and $\mathbf{w} = \begin{bmatrix} 1 & -1 & c \end{bmatrix}^T$ where $c \in \mathbb{R}$. Find the value(s) of $c$ such that the set $\mathcal{B} := \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ form a basis of $\mathbb{R}^3$.

Ans. First remark that if $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ are linearly independent then they must span $\mathbb{R}^3$ as $\dim(\mathbb{R}^3) = 3$. So any set of three linearly independent vectors will form a basis of $\mathbb{R}^3$ (try to verify this). Therefore, it is sufficient to find $c$ such that the collection $\mathcal{B}$ is linearly independent. Toward this end, we consider the linear combination

$$c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} = \mathbf{0}, \tag{1}$$

and solve it for $c_1, c_2$, and $c_3$. This can be equivalently written as

$$\begin{pmatrix} 2 & 3 & 1 \\ 0 & 1 & -1 \\ -1 & 0 & c \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We find the echelon form of the coefficient matrix as follows

$$\begin{aligned}
\begin{pmatrix} 2 & 3 & 1 \\ 0 & 1 & -1 \\ -1 & 0 & c \end{pmatrix} &\sim \begin{pmatrix} 1 & 3/2 & 1/2 \\ 0 & 1 & -1 \\ -1 & 0 & c \end{pmatrix} && (R_1 \to \tfrac{1}{2}R_1) \\
&\sim \begin{pmatrix} 1 & 3/2 & 1/2 \\ 0 & 1 & -1 \\ 0 & 3/2 & c+1/2 \end{pmatrix} && (R_3 \to R_3 + R_1) \\
&\sim \begin{pmatrix} 1 & 3/2 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & c+2 \end{pmatrix} && (R_3 \to R_3 - \tfrac{3}{2}R_2).
\end{aligned}$$

Note that for $c + 2 = 0$ the coefficient matrix will be rank deficient and constant $c_3$ will become a free variable. Thus, there will be non-trivial solutions to the system (1). Thus, the collection $\mathcal{B}$ is linearly independent and hence, forms a basis of $\mathbb{R}^3$ if and only if $c \neq -2$.

Q.4 Define a transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ by

$$T(\mathbf{x}) = T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} := \begin{pmatrix} x_1 - x_3 \\ x_1 + x_2 \\ x_3 - x_2 \\ x_1 - 2x_2 \end{pmatrix}, \qquad \text{for all } \mathbf{x} \in \mathbb{R}^3.$$

(a) Find $T(\mathbf{x})$ for $\mathbf{x}^T := \begin{pmatrix} 1 & -2 & 3 \end{pmatrix}$.

Ans. It is easy to calculate

$$T(\mathbf{x}) = T\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1-3 \\ 1+(-2) \\ 3-(-2) \\ 1-2(-2) \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 5 \\ 5 \end{pmatrix}.$$

(b) Show that $T$ is a linear transformation.

Ans. We verify that $T(c\mathbf{x}) = cT(\mathbf{x})$ for all $c \in \mathbb{R}$ and for all $\mathbf{x} \in \mathbb{R}^3$, and $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$. We have, for any $\mathbf{x} \in \mathbb{R}^3$ and for any $c \in \mathbb{R}^3$,

$$T(c\mathbf{x}) = T\left(c\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = T\begin{pmatrix} cx_1 \\ cx_2 \\ cx_3 \end{pmatrix} = \begin{pmatrix} cx_1 - cx_3 \\ cx_1 + cx_2 \\ cx_3 - cx_2 \\ cx_1 - 2cx_2 \end{pmatrix} = c\begin{pmatrix} x_1 - x_3 \\ x_1 + x_2 \\ x_3 - x_2 \\ x_1 - 2x_2 \end{pmatrix} = cT(\mathbf{x}).$$

Moreover, let $\mathbf{x} = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}^T, \mathbf{y} = \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix}^T \in \mathbb{R}^3$ be two arbitrary vectors. Then

$$\begin{aligned} T(\mathbf{x}+\mathbf{y}) &= \left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right) = T\begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{pmatrix} = \begin{pmatrix} (x_1+y_1) - (x_3+y_3) \\ (x_1+y_1) + (x_2+y_2) \\ (x_3+y_3) - (x_2+y_2) \\ (x_1+y_1) - 2(x_2+y_2) \end{pmatrix} \\ &= \begin{pmatrix} x_1 - x_3 \\ x_1 + x_2 \\ x_3 - x_2 \\ x_1 - 2x_2 \end{pmatrix} + \begin{pmatrix} y_1 - y_3 \\ y_1 + y_2 \\ y_3 - y_2 \\ y_1 - 2y_2 \end{pmatrix} = T(\mathbf{x}) + T(\mathbf{y}). \end{aligned}$$

(c) Find the matrix $A$ of transformation $T$ such that $T(\mathbf{x}) = A\mathbf{x}$.

Ans. Apparently, the matrix of transformation, $A$, is given by

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix}.$$

A more systematic way is to use the standard (usual) basis $\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$ of $\mathbb{R}^3$ as follows. Since,

$$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \\ -2 \end{pmatrix}, \quad T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Therefore, the transformation matrix of $T$ is given by

$$A = \begin{pmatrix} T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} & T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} & T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix}.$$

(d) The KERNEL of aforementioned $T : \mathbb{R}^3 \to \mathbb{R}^4$, denoted by $\ker(T)$, is defined to be **the set of all $\mathbf{x}$ in $\mathbb{R}^3$ such that $T\mathbf{x} = \mathbf{0}$**. It is also called the *null space* of $T$. Find $\ker(T)$.

Ans. In order to find $\ker(T)$, we find all $\mathbf{x} \in \mathbb{R}^3$ such that $T(\mathbf{x}) = \mathbf{0}$. From the part (c), we know that $T(\mathbf{x}) = A\mathbf{x}$. Therefore, $\ker(T)$ is the solution set of the homogeneous system $A\mathbf{x} = \mathbf{0}$. Towards this end, we reduce the matrix $A$ to its reduced echelon form as follows.

$$
\begin{aligned}
A &= \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix} \\
&\sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & -2 & 1 \end{pmatrix}, \qquad (R_2 \to R_2 - R_1, \quad R_4 \to R_4 - R_1) \\
&\sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{pmatrix}, \qquad (R_3 \to R_3 + R_2, \quad R_4 \to R_4 + 2R_2) \\
&\sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (R_3 \to \frac{1}{2}R_3, \quad R_4 \to \frac{1}{3}R_4) \\
&\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad (R_4 \to R_4 - R_3, \quad R_1 \to R_1 + R_3, \quad R_2 \to R_2 - R_3).
\end{aligned}
$$

Therefore, we have $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$ as the unique solution to the system $A\mathbf{x} = \mathbf{0}$. Therefore, the $\ker(T) = \{\mathbf{0}\}$.

(e) Show that the $\ker(T)$ derived in part (d) is a subspace of $\mathbb{R}^3$.

Ans. From part (d), it is clear that $\ker(T) = \{\mathbf{0}\} \subset \mathbb{R}^3$, which is a trivial subspace of $\mathbb{R}^3$. There is nothing to prove.

(f) Find $\dim(\ker(T))$, i.e., the dimension of subspace $\ker(T)$. *(Hint: First express* $\ker(T) := \text{span}\{\mathbf{v}_1, \cdots\}$ *for some vector(s), then show that the vector(s)* $\mathbf{v}_1, \cdots$ *are linearly independent.)*

Ans. It is clear from parts (d) and (e) that the $\dim(\ker(T)) = 0$, since there is no non-zero vector in $\ker(T)$.

(g) The RANGE of aforementioned $T : \mathbb{R}^3 \to \mathbb{R}^4$, denoted by $\text{rang}(T)$, is **the set of all $\mathbf{b} \in \mathbb{R}^4$ such that $T(\mathbf{x}) = \mathbf{b}$ for some $\mathbf{x} \in \mathbb{R}^3$**. Find $\text{rang}(T)$.

Ans. In order to find $\text{rang}(T)$, we identify all $\mathbf{b} \in \mathbb{R}^4$ for which there exists at least one $\mathbf{x} \in \mathbb{R}^3$ such that $T(\mathbf{x}) = \mathbf{b}$. As $T(\mathbf{x}) = A\mathbf{x}$ from part(c), we actually solve the non-homogeneous system $A\mathbf{x} = \mathbf{b}$ for arbitrary $\mathbf{b}$ and identify all vectors $\mathbf{b}$ for which the system $A\mathbf{x} = \mathbf{b}$ is consistent. Consider $\mathbf{b} = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \end{pmatrix}^T \in \mathbb{R}^4$ arbitrary. Then the augmented matrix of the non-homogeneous system is

$$[A \,|\, \mathbf{b}] = \begin{pmatrix} 1 & 0 & -1 & b_1 \\ 1 & 1 & 0 & b_2 \\ 0 & -1 & 1 & b_3 \\ 1 & -2 & 0 & b_4 \end{pmatrix}.$$

We perform Gaussian elimination method on the augmented matrix as follows,

$$\begin{aligned}
[A \,|\, \mathbf{b}] &\sim \begin{pmatrix} 1 & 0 & -1 & b_1 \\ 0 & 1 & 1 & -b_1 + b_2 \\ 0 & -1 & 1 & b_3 \\ 0 & -2 & 1 & -b_1 + b_4 \end{pmatrix}, && (R_2 \to R_2 - R_1, \quad R_4 \to R_4 - R_1) \\
&\sim \begin{pmatrix} 1 & 0 & -1 & b_1 \\ 0 & 1 & 1 & -b_1 + b_2 \\ 0 & 0 & 2 & -b_1 + b_2 + b_3 \\ 0 & 0 & 3 & -3b_1 + 2b_2 + b_4 \end{pmatrix}, && (R_3 \to R_3 + R_2, \quad R_4 \to R_4 + 2R_2) \\
&\sim \begin{pmatrix} 1 & 0 & -1 & b_1 \\ 0 & 1 & 1 & -b_1 + b_2 \\ 0 & 0 & 1 & (-b_1 + b_2 + b_3)/2 \\ 0 & 0 & 1 & (-3b_1 + 2b_2 + b_4)/3 \end{pmatrix}, && (R_3 \to \frac{1}{2}R_3, \quad R_4 \to \frac{1}{3}R_4) \\
&\sim \begin{pmatrix} 1 & 0 & -1 & b_1 \\ 0 & 1 & 1 & -b_1 + b_2 \\ 0 & 0 & 1 & (-b_1 + b_2 + b_3)/2 \\ 0 & 0 & 0 & (-3b_1 + b_2 - 3b_3 + 2b_4)/6 \end{pmatrix}, && (R_4 \to R_4 - R_3)
\end{aligned}$$

Therefore, for any vector $\mathbf{b} = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \end{pmatrix}^T \in \mathbb{R}^4$ to be in range of $T$, we must have

$$-3b_1 + b_2 - 3b_3 + 2b_4 = 0 \quad \text{or equivalently} \quad b_4 = (3b_1 - b_2 + 3b_3)/2, \tag{2}$$

otherwise the system $A\mathbf{x} = \mathbf{b}$ is inconsistent. Therefore,

$$\text{rang}(T) = \left\{ \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; b_1, b_2, b_3 \in \mathbb{R} \text{ are arbitrary and } b_4 = \frac{(3b_1 - b_2 + 3b_3)}{2} \right\}.$$

(h) Show that rang$(T)$ is a subspace of $\mathbb{R}^4$.

Ans. In order to prove that rang$(T) \subset \mathbb{R}^4$ is a subspace of $\mathbb{R}^4$, we verify three properties.

1. $\mathbf{0} \in \mathbb{R}^4$ belongs to rang$(T)$. Indeed, for $b_1 = 0$, $b_2 = 0$, and $b_3 = 0$, we have $b_4 = (3(0) - (0) + 3(0))/2 = 0$.
2. Let $\mathbf{u} := \begin{pmatrix} u_1 & u_2 & u_3 & u_4 \end{pmatrix}^T \in \text{rang}(T)$ be any vector. Then, by definition $u_4 = (3u_1 - u_2 + 3u_3)/2$. For arbitrary $c \in \mathbb{R}$, the vector $c\mathbf{u} := \begin{pmatrix} cu_1 & cu_2 & cu_3 & cu_4 \end{pmatrix}^T$ belongs to rang$(T)$ since we have $cu_4 = (3(cu_1) - (cu_2) + 3(cu_3))/2$, and thus, the vector $c\mathbf{u}$ satisfies (2).
3. Let $\mathbf{u} = \begin{pmatrix} u_1 & u_2 & u_3 & u_4 \end{pmatrix}^T, \mathbf{v} = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \end{pmatrix}^T \in \text{rang}(T)$ be arbitrary. Then $u_4 = (3u_1 - u_2 + 3u_3)/2$ and $v_4 = (3v_1 - v_2 + 3v_3)/2$. Note that

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 + v_1 & u_2 + v_2 & u_3 + v_3 & u_4 + v_4 \end{pmatrix}^T.$$

Moreover,

$$u_4 + v_4 = (3u_1 - u_2 + 3u_3)/2 + (3v_1 - v_2 + 3v_3)/2 = (3(u_1 + v_1) - (u_2 + v_2) + 3(u_3 + v_3))/2.$$

Therefore, $\mathbf{u} + \mathbf{v}$ vector satisfies (2) and thus, $\mathbf{u} + \mathbf{v} \in \text{rang}(T)$.

(i) Find the dim(rang$(T)$) following a similar procedure as in part (f).

Ans. Note that the range space of $T$ can be expressed as

$$\text{rang}(T) = \left\{ \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ (3b_1 - b_2 + 3b_3)/2 \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; b_1, b_2, b_3 \in \mathbb{R} \text{ are arbitrary} \right\}.$$

Since $b_1, b_2, b_3$ are arbitrary, any vector in rang$(T)$ can be expressed as

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ (3b_1 - b_2 + 3b_3)/2 \end{pmatrix} = b_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 3/2 \end{pmatrix} + b_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1/2 \end{pmatrix} + b_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3/2 \end{pmatrix}.$$

Therefore,

$$\text{rang}(T) = \text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}, \quad \text{where } \mathbf{u} := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 3/2 \end{pmatrix}, \quad \mathbf{v} := \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1/2 \end{pmatrix} \text{ and } \mathbf{w} := \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3/2 \end{pmatrix}.$$

It only remains to verify that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ are linearly independent. Consider the system

$$c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} = \mathbf{0} \quad \text{or equivalently} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3/2 & -1/2 & 3/2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Row reduction of the coefficient matrix renders

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3/2 & -1/2 & 3/2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1/2 & 3/2 \end{pmatrix} \qquad (R_4 - \frac{3}{2}R_1)$$

$$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 3/2 \end{pmatrix} \qquad (R_4 + \frac{1}{2}R_2)$$

$$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad (R_4 - \frac{3}{2}R_3).$$

Therefore, we have $c_1 = 0$, $c_2 = 0$ and $c_3 = 0$. Thus, $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent. As $\text{rang}(T) = \text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$. Therefore, $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ forms a basis of $\text{rang}(T)$. Hence, $\dim(\text{rang}(T)) = 3$.

(j) Verify that $\dim(\text{rang}(T)) + \dim(\ker(T)) = \dim(\mathbb{R}^3)$? What do you conclude from this for a general linear transformation $T_2 : \mathbb{R}^n \to \mathbb{R}^m$, for $m, n \in \mathbb{N}$?

Ans. It is evident that $\dim(\text{rang}(T)) + \dim(\ker(T)) = 0 + 3 = \dim(\mathbb{R}^3) = 3$. It can be concluded that $\dim(\text{rang}(T)) + \dim(\ker(T)) = n = \dim(\mathbb{R}^n)$. This is, in fact, the so-called *Rank-Nullity Theorem*.

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**"Every stumble is not a fall, and every fall does not mean failure." ∼ Oprah Winfrey**