National University of Technology, Islamabad
Assignment II (Linear Algebra and ODE), Fall 2019
Due Date: Oct. 18, 2019

Q.1 Consider the matrix $\begin{bmatrix} 1 & 3 & 2 \\ a & 6 & 2 \\ 0 & 9 & 5 \end{bmatrix}$ where a is a real number. For what value of a is the matrix singular?

Q.2 Find the inverse of $\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}$.

Q.3 Recall that a set $\mathcal{B}$ of vectors of a subspace $\mathcal{S}$ of $\mathbb{R}^n$ is called a *basis* of $\mathcal{S}$ if $\mathcal{B}$ is linearly independent and it spans $\mathcal{S}$. Let $\mathbf{u} = \begin{bmatrix} 2 & 0 & -1 \end{bmatrix}^T$, $\mathbf{v} = \begin{bmatrix} 3 & 1 & 0 \end{bmatrix}^T$, and $\mathbf{w} = \begin{bmatrix} 1 & -1 & c \end{bmatrix}^T$ where $c \in \mathbb{R}$. Find the value(s) of c such that the set $\mathcal{B} := \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ form a basis of $\mathbb{R}^3$.

Q.4 Define a transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ by

$$T(\mathbf{x}) = T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} := \begin{pmatrix} x_1 - x_3 \\ x_1 + x_2 \\ x_3 - x_2 \\ x_1 - 2x_2 \end{pmatrix}, \qquad \text{for all } \mathbf{x} \in \mathbb{R}^3.$$

(a) Find $T(\mathbf{x})$ for $\mathbf{x}^T := \begin{pmatrix} 1 & -2 & 3 \end{pmatrix}$.

(b) Show that T is a linear transformation.

(c) Find the matrix A of transformation T such that $T(\mathbf{x}) = A\mathbf{x}$.

(d) The Kernel of aforementioned $T : \mathbb{R}^3 \to \mathbb{R}^4$, denoted by $\ker(T)$, is defined to be **the set of all $\mathbf{x}$ in $\mathbb{R}^3$ such that $T\mathbf{x} = \mathbf{0}$**. It is also called the *null space* of T. Find $\ker(T)$.

(e) Show that the $\ker(T)$ derived in part (d) is a subspace of $\mathbb{R}^3$.

(f) Find $\dim(\ker(T))$, i.e., the dimension of subspace $\ker(T)$. *(Hint: First express* $\ker(T) := \text{span}\{\mathbf{v}_1, \cdots\}$ *for some vector(s), then show that the vector(s)* $\mathbf{v}_1, \cdots$ *are linearly independent.)*

(g) The Range of aforementioned $T : \mathbb{R}^3 \to \mathbb{R}^4$, denoted by $\text{rang}(T)$, is **the set of all $\mathbf{b} \in \mathbb{R}^4$ such that $T(\mathbf{x}) = \mathbf{b}$ for some $\mathbf{x} \in \mathbb{R}^3$**. Find $\text{rang}(T)$.

(h) Show that $\text{rang}(T)$ is a subspace of $\mathbb{R}^4$.

(i) Find the $\dim(\text{rang}(T))$ following a similar procedure as in part (f).

(j) Verify that $\dim(\text{rang}(T)) + \dim(\ker(T)) = \dim(\mathbb{R}^3)$? What do you conclude from this for a general linear transformation $T_2 : \mathbb{R}^n \to \mathbb{R}^m$, for $m, n \in \mathbb{N}$?

"Every stumble is not a fall, and every fall does not mean failure." ∼ Oprah Winfrey