National University of Technology, Islamabad
Assignment IV (Calculus II), Spring 2019
Solution Key

Q.1 Assume that the trajectory is represented parametrically by the equations

$$x = x(t) \quad \text{and} \quad y = y(t), \quad \text{i.e.,} \quad \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j},$$

where the particle is at the point $(1, 4)$ at time $t = 0$ (i.e., $x(0) = 1$ and $y(0) = 4$). Because the particle moves in the direction of maximum temperature increase, its direction of motion at time t is in the direction of the gradient of $T(x, y)$ and hence its velocity vector $\mathbf{v}(t)$ at time t points in the direction of the gradient. Thus, there is a scalar k that depends on t such that

$$\mathbf{r}'(t) = \mathbf{v}(t) = k\nabla T(x, y) = -8kx\mathbf{i} - 2ky\mathbf{j}.$$

Since $\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j}$, this implies

$$\frac{dx}{dt} = -8kx \quad \text{and} \quad \frac{dy}{dt} = -2ky.$$

Therefore,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y}{4x}.$$

This is a separable first order ordinary differential equations. One separation of variables, we obtain

$$4\int \frac{dy}{y} = \frac{dx}{x} \implies 4\ln|y| = ln|x| + \ln|c| \implies y^4 = cx.$$

Since $y(1) = 4$ (the particle starts its trajectory from the point $(1, 4)$), therefore, $256 = c(1) = c$. Thus, $y^4 = 256x$. The particle travels along the curve $y^4 = 256x$. By solving $dx/dt = -8kx$ and $dy/dt = -2ky$ in the similar fashion, we can find that one possible parametrization of the trajectory of the particle is $x(t) = e^{-8t}$, $y(t) = e^{-2t}$.

Q.2 Let x, y, z, and S be the length, width, height (in ft.), and surface area of the box (in sq. ft.), respectively. We may reasonably assume that the box with least surface area requires the least amount of material, so our objective is to minimize the surface area

$$S = xy + 2xz + 3yz \quad \text{subject to the volume requirement} \quad xyz = 32ft^3.$$

Thanks to the constrain, we can eliminate z from the expression of S, i.e., by replacing $z = 32/xy$ in the surface area, we obtain

$$S(x, y) = xy + \frac{64}{x} + \frac{64}{y}.$$

The dimensions x and y in this formula must be positive, but otherwise have no limitations, so our problem reduces to finding the absolute minimum value of S over the open first quadrant: $x > 0$ and $y > 0$. Because this region is neither closed nor bounded, we have no mathematical guarantee at this stage that an absolute minimum exists. However, if S has an absolute minimum value in the open first quadrant, then it must occur at a critical point of S. Thus, our next step is to find the critical points of S. Towards this end, we have

$$\frac{\partial S}{\partial x} = y - \frac{64}{x^2} \quad \text{and} \quad \frac{\partial S}{\partial y} = x - \frac{64}{y^2},$$

so the coordinates of the critical points of S satisfy

$$y - \frac{64}{x^2} = 0 \quad \text{and} \quad x - \frac{64}{y^2} = 0.$$

Solving the first equation for y and substituting the expression in the second equation yields

$$x - \frac{64}{(64/x^2)^2} = 0 \iff x\left(1 - \frac{x^3}{64}\right) = 0.$$

The solutions of this equations are $x = 0$ and $x = 4$. Since we require $x > 0$, the only solution of significance is $x = 4$. Substituting this value into $y = 64/x^2$ yields $y = 4$. We conclude that the point $(x, y) = (4, 4)$ is the only critical point of S in the first quadrant. Since $S = 48$ if $x = 4 = y$, this suggests we try to show that the minimum value of S on the open first quadrant is 48.

It immediately follows from the expression of S that $48 < S$ at any point in the first quadrant for which at least one of the inequalities $xy > 8$, $64/y > 48$, and $64/x > 48$ is satisfied. Therefore, to prove that $48 \leq S$, we restrict attention to the set of points in the first quadrant that satisfy the three inequalities $xy \leq 48$, $64/y \leq 48$ and $64/x \leq 48$. These inequalities can be rewritten as

$$xy \leq 48, \qquad y \geq \frac{4}{3}, \qquad x \geq \frac{4}{3},$$

and they define a closed and bounded region R within the first quadrant. The function S is continuous on R, that we can guarantee that S has an absolute minimum value somewhere on R. Since the point $(4, 4)$ lies within R, and $48 < S$ on the boundary of R, the minimum value of S on R must occur at an interior point. It then follows that the minimum of S on R must occur at a critical point of S. Hence, the absolute minimum of S on R (and therefore on the entire open first quadrant) is $S = 48$ at the point $(4, 4)$. Substituting $x = 4$ and $y = 4$ furnishes $z = 32/16ft = 2ft$, so the box using the least material has a height of $2ft$ and a square base whose edges are 4 ft long.

Q.3 We need to minimize $S = x^2 + y^2 + z^2$ subject to $x + y + z = 27$, $x > 0$, $y > 0$ and $z > 0$. We eliminate z by substituting $z = 27 - x - y$ in the expression of S so that

$$S(x, y) = x^2 + y^2 + (27 - x - y)^2.$$

For critical points we set $S_x = 4x + 2y - 54 = 0$ and $S_y = 2x + 4y - 54 = 0$. Solving these two simultaneous equations, we get the only critical point $(9, 9)$. Note that $S_{xx} = 4$ $S_{yy} = 4$ and $S_{xy} = 0$. Thus, $H = S_{xx}S_{yy} - S_{xy}^2 = 16 > 0$. Since $S_{xx} = 4 > 0$ at $(9, 9)$, there is a relative minimum of S at $(9, 9)$. For $x = 9 = y$, we have $z = 27 - x - y = 9$. Therefore, the sum of the squares is minimum for the numbers $x = 9$, $y = 9$, and $z = 9$ together with $x + y + z = 27$.

Q.4 We need to maximize the profit $P = 500(y - x)(x - 40) + [45000 + 500(x - 2y)](y - 60) = 500(-x^2 - 2y^2 + 2xy - 20x + 170y - 5400)$. As $P_x = 1000(-x + y - 10)$, $P_y = 1000(-2y + x + 85)$; setting $P_x = 0 = P_y$, we get the critical point $(65, 75)$. Note that $P_{xx} = -1000$, $P_{yy} = -2000$ and $P_{xy} = 0$, we have $H = P_{xx}P_{yy} - P_{xy}^2 > 0$ and $P_{xx} < 0$. Therefore, the profit is relative maximum at $(65, 75)$. There profit will be maximum when $x = 65$ and $y = 75$.

Q.5 We minimize the square of the distance $w = x^2 + y^2 + z^2$ of the closest point (x, y, z) (say) on the surface $x^2 - yz = 5$ to the origin $(0, 0, 0)$. That is, we minimize $w = x^2 + y^2 + z^2$ subject to $x^2 - yz = 5$. If we eliminate x by substituting $x^2 = 5 + yz$ in the expression of w, we get

$$w(y, z) = 5 + yz + y^2 + z^2.$$

Note that $w_y = z + 2y$ and $w_z = y + 2z$. Setting $w_z = 0 = w_y$ renders $y = 0 = z$. Thus, the only critical point is $(0, 0)$. Since $w_{yy}w_{zz} - w_{yz}^2 = 3 > 0$ and $w_{yy} = 2 > 0$, function w has relative minimum at $y = 0$ and $z = 0$ and $x^2 = 5 - yz = 5$. Thus, the points $(\pm\sqrt{5}, 0, 0)$ on the surface $x^2 - yz = 5$ are closest to the origin.

Q.6 We maximize $w = xy^2z^2$ subject to $x + y + z = 5$, $x > 0$, $y > 0$, and $z > 0$. Substituting $x = 5 - y - z$ we obtain

$$w(y, z) = (5 - y - z)y^2z^2 = 5y^2z^2 - y^3z^2 - y^2z^3.$$

Therefore, $w_y = 10yz^2 - 3y^2z^2 - 2yz^3 = yz^2(10 - 3y - 2z)$ and $w_z = 10y^2z - 2y^3z - 3y^2z^2 = y^2z(10 - 2y - 3z)$. Setting $w_y = 0 = w_z$, we obtain the critical point $y = 2$, $z = 2$ and thus $x = 1$. Note that $w_{yy}w_{zz} - w_{yz}^2 = 320 > 0$ and $w_{yy} = -24 < 0$ when at $y = 2 = z$. Therefore, w has relative maximum when $x = 1$, $y = 2$ and $z = 2$. Thus, xy^2z^2 is maximum at $(1, 2, 2)$.

"There are two types of people who will tell you that you cannot make a difference in this world: those who are afraid to try and those who are afraid you will succeed." ~ Ray Goforth