National University of Technology, Islamabad
Assignment V (Calculus II), Spring 2019
Solution Key

Q.1 Here, the constraint is $g(x,y) = 4x^2 + y^2 - 4 = 0$. Setting $\nabla f(x,y) = \lambda \nabla g(x,y)$, we obtain

$$y\mathbf{i} + x\mathbf{j} = \lambda(8x\mathbf{i} + 2y\mathbf{j}).$$

Comparing the components on both sides of the above equation and together with the constraint, we obtain the system

$$y = 8x\lambda, \qquad x = 2y\lambda, \qquad 4x^2 + y^2 = 4.$$

There are many ways to solve above system for x, y and λ. One way is by first eliminating y from $x = 2y\lambda$ as

$$x = 2y\lambda = 2(8x\lambda)\lambda = 16x\lambda^2 \implies x(1 - 16\lambda^2) = 0 \implies x = 0 \text{ or } \lambda = \pm 1/4.$$

If $x = 0$, then $4x^2 + y^2 = 4$ furnishes $y = \pm 2$. Thus, the points $(0, \pm 2)$ are possibilities for extrema of $f(x,y)$. If $\lambda = \pm 1/4$, then $y = 8x(\pm 1/4) = \pm 2x$. Again using fact that $4x^2 + y^2 = 4$ we get

$$4x^2 + 4x^2 = 4 \implies 8x^2 = 4 \implies x = \pm 1/\sqrt{2}.$$

The corresponding $y-$values are therefore $y = \pm 2x = \pm 2/\sqrt{2} = \pm\sqrt{2}$. This gives us the points $(\sqrt{2}/2, \pm\sqrt{2})$ and $(-\sqrt{2}/2, \pm\sqrt{2})$. The values of f at each of the points we have found are

$$f(0,2) = 0 = f(0,-2), \quad f(\frac{\sqrt{2}}{2}, \sqrt{2}) = 1 = f(-\frac{\sqrt{2}}{2}, -\sqrt{2}), \quad f(-\frac{\sqrt{2}}{2}, \sqrt{2}) = -1 = f(\frac{\sqrt{2}}{2}, -\sqrt{2}).$$

It follows that $f(x,y)$ takes on a maximum value 1 at either $(\frac{\sqrt{2}}{2}, \sqrt{2})$ or $(-\frac{\sqrt{2}}{2}, -\sqrt{2})$ and minimum value 1 at $(-\frac{\sqrt{2}}{2}, \sqrt{2})$ or $(\frac{\sqrt{2}}{2}, -\sqrt{2})$.

Q.2 Let $2x$, $2y$, and $2z$ be the length (from $-x$ to x along x-axis), width (from $-y$ to y along y-axis), and height (from $-z$ to z along z-axis)of the box. We wish to maximize $V = f(x,y,z) = 8xyz$ subject to the constraint $g(x,y,z) = 16x^2 + 4y^2 + 9z^2 - 144 = 0$. Let us begin by considering $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$, or

$$8yz\mathbf{i} + 8xz\mathbf{j} + 8xy\mathbf{k} = \lambda\left(32x\mathbf{i} + 8y\mathbf{j} + 18z\mathbf{k}\right).$$

Comparing the components on both sides of the above equation, together with $g(x,y,z) = 0$ gives us the system of four equations

$$8yz = 32x\lambda, \quad 8xz = 8y\lambda, \quad 8xy = 18z\lambda, \quad 16x^2 + 4y^2 + 9z^2 = 144.$$

Multiplying the first equations by x, the second by y, and the third by z and adding gives us

$$24xyz = 32x^2\lambda + 8y^2\lambda + 18z^2\lambda = 2\lambda(16x^2 + 4y^2 + 9z^2) = 2(144)\lambda \implies xyz = 12\lambda.$$

The last equation may be used to find x, y, and z. For example, multiplying both sides of the equation $8yz = 32x\lambda$ by x, we obtain

$$8xyz = 32x^2\lambda \implies 8(12)\lambda = 32x^2\lambda \implies 96\lambda - 32x^2\lambda = 0 \implies \lambda(3 - x^2) = 0.$$

Consequently, either $\lambda = 0$ or $x = \sqrt{3}$ (remember that $x > 0$ is a length). Moreover, we may reject $\lambda = 0$ since in this case $xyz = 0$ and hence $V = 8xyz = 0$. Thus the only possibility for x is $\sqrt{3}$.

Similarly, multiplying both sides of the equation $8xz = 8y\lambda$ by y leads to

$$8xyz = 8y^2\lambda \implies 8(12\lambda) = 8y^2\lambda \implies \lambda(12 - y^2) = 0 \implies y = \sqrt{12} = 2\sqrt{3}.$$

Finally, multiplying the equation $8xy = 18z\lambda$ by z and using the same technique results in $z = 4/\sqrt{3}$. It follows that the desired volume is

$$V = 8xyz = 8(\sqrt{3})(2/\sqrt{3})(4/\sqrt{3}) = 64\sqrt{3}.$$

Q.3 The objective function (square of the distance function) here is

$$f(x, y, z) = (x - 1)^2 + (y + 1)^2 + (z - 1)^2.$$

and the constraint is $g(x, y, z) = 4x + 3y + z - 2 = 0$. For Lagrange multipliers, we set the equation

$$\nabla f = \lambda \nabla g \iff 2(x - 1)\mathbf{i} + 2(y + 1)\mathbf{j} + 2(z - 1)\mathbf{k} = \lambda\left(4\mathbf{i} + 3\mathbf{j} + \mathbf{k}\right).$$

This together with the constraint gives the equations

$$2(x - 1) = 4\lambda, \quad 2(y - 1) = 3\lambda, \quad 2(z - 1) = \lambda, \quad 4x + 3y + z = 2.$$

Substituting $\lambda = 2z - 2$ in the first, second, and the last equation above, we get

$$x = 2\lambda + 1 = 4z - 4 + 1 = 4z - 3, \quad y = \frac{3}{2}\lambda - 1 = 3z - 3 + 1 = 3z - 4.$$

and

$$2 = 4x + 3y + z = 4(4z - 3) + 3(3z - 4) + z = 26z - 24 \iff 26z = 26 \iff z = 1.$$

Therefore, $x = 4(1) - 3 = 1$, $y = 3(1) - 4 = -1$, $z = 1$. Thus the point is $(1, -1, 1)$. That was expected because point $(1, -1, 1)$ lies on the plane $4x + 3y + z = 2$.

Q.4 Let $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be the required vector such that $|\mathbf{v}| = 5$, i.e., $x^2 + y^2 + z^2 = 25$ and $f(x, y, z) = x + y + z$ is largest (f is the objective function here). The problem at hand is as follows. Maximize the function $f(x, y, z) = x + y + z$ subject to the constrain $g(x, y, z) = x^2 + y^2 + z^2 - 25$. Using the method of Lagrange multipliers, we set $\nabla f = \lambda \nabla g$, i.e.,

$$\mathbf{i} + \mathbf{j} + \mathbf{k} = \lambda(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}) \iff 1 = 2x\lambda, \quad 1 = 2y\lambda, \quad 1 = 2z\lambda.$$

This implies

$$\lambda = \frac{1}{2x} = \frac{1}{2y} = \frac{1}{2z} \iff y = x \quad \text{and} \quad z = x.$$

With this information at hand, we have from the constraint equation

$$25 = x^2 + y^2 + z^2 = x^2 + x^2 + x^2 = 3x^2 \iff x = \pm\sqrt{25/3} = \pm 5/\sqrt{3} = y = z.$$

Since

$$f\left(\frac{5}{\sqrt{3}}, \frac{5}{\sqrt{3}}, \frac{5}{\sqrt{3}}\right) = \frac{5}{\sqrt{3}} \quad \text{and} \quad f\left(-\frac{5}{\sqrt{3}}, -\frac{5}{\sqrt{3}}, -\frac{5}{\sqrt{3}}\right) = -\frac{5}{\sqrt{3}},$$

therefore, the required vector is $\mathbf{v} = \frac{5}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k})$.

Q.5 The ant is constrained to traverse in a circle of radius 5. Therefore, assuming the center of the circle to be the origin, the constraint equation is $x^2 + y^2 = 25$. Thus, the objective function is $T(x, y) = 4x^2 - 4xy$ and the constraint function is $g(x, y) = x^2 + y^2 - 25$. By using method of Lagrange multipliers, we set $\nabla T = \lambda \nabla g$, i.e.,

$$(8x - 4y)\mathbf{i} - 4x\mathbf{j} = \lambda(2x\mathbf{i} + 2y\mathbf{j}).$$

In components, we have $8x - 4y = 2x\lambda$ and $-4x + 2y = 2y\lambda$. Note that x and y cannot be zero, because if $x = 0$ then $y = 0$ and conversely; however, $x^2 + y^2 = 25$. Therefore, both x and y are non-zero. Thus, $\lambda = (4x - 2y)/x$ and $\lambda = (-2x + y)/y$ from the component equations above. So

$$(4x - 2y)/x = (-2x + y)/y \quad \iff \quad 2x^2 + 3xy - 2y^2 = 0 \quad \iff \quad (2x - y)(x + 2y) = 0.$$

Therefore, either $y = 2x$ or $x = -2y$. If $y = 2x$ then $x^2 + (2x)^2 = 25$, i.e., $x = \pm\sqrt{5}$. If $x = -2y$ then $(-2y)^2 + y^2 = 25$, i.e., $y = \pm\sqrt{5}$. Therefore, the possible points of extrema are $(-\sqrt{5}, -2\sqrt{5})$, $(\sqrt{5}, 2\sqrt{5})$, $(-2\sqrt{5}, \sqrt{5})$ and $(2\sqrt{5}, -\sqrt{5})$. Since

$$T(-\sqrt{5}, -2\sqrt{5}) = 0 = T(\sqrt{5}, 2\sqrt{5}) \quad \text{and} \quad T(-2\sqrt{5}, \sqrt{5}) = 125 = T(2\sqrt{5}, -\sqrt{5}).$$

The highest temperature is 125 and the lowest temperature is 0 subject to the constraint $x^2 + y^2 = 25$.

"The two most important days in your life are the day you are born and the day you find out why." — Mark Twain