National University of Technology, Islamabad
Assignment VI (Calculus II), Spring 2019
Solution Key

Q.1

a. $\int_0^{\pi/2}\int_0^1 y\sin x dydx = \int_0^{\pi/2}\left[\frac{y^2}{2}\sin x\right]_0^1 dx = \int_0^{\pi/2}\frac{1}{2}\sin x = \left[-\frac{1}{2}\cos x\right]_0^{\pi/2} = \frac{1}{2}.$

b. $\int_0^2\int_{x^2}^x y^2 xdydx = \int_0^2\left[\frac{y^3x}{3}\right]_{x^2}^x dx = \int_0^2\left[\frac{x^4}{3}-\frac{x^7}{3}\right]dx = \left[\frac{x^5}{15}-\frac{x^8}{24}\right]_0^2 = -\frac{128}{15}.$

c. $\int_{\pi/2}^{\pi}\int_0^{x^2}\frac{1}{x}\cos\left(\frac{y}{x}\right)dydx = \int_{\pi/2}^{\pi}\left[\frac{1}{x}\frac{\sin\left(\frac{y}{x}\right)}{\frac{1}{x}}\right]_0^{x^2}dx = \int_{\pi/2}^{\pi}\sin xdx = [-\cos x]_{\pi/2}^{\pi} = 1.$

Q.2 The plane $3x+2y+z=12$ is a function $z=12-3x-2y$, so the volume of the solid is

$$\begin{aligned}V &= \iint_R zdA = \int_{-2}^3\int_0^1(12-3x-2y)dxdy = \int_{-2}^3\left[12x-\frac{3}{2}x^2-2xy\right]_0^1 dy\\ &= \int_{-2}^3\left(\frac{21}{2}-2y\right)dy = \left[\frac{21}{2}y-y^2\right]_{-2}^3 = \frac{95}{2}.\end{aligned}$$

Q.3 The graphs of $y=\sqrt{x}$ and $y=\sqrt{3x-18}$ are the top halves of the parabolas $y^2=x$ and $y^2=3x-18$. The region R is sketched in Figure 1. If we wish to use vertical slicing then

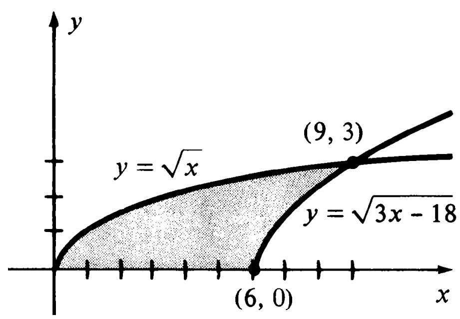

Figure 1: The region of integration in $Q.3$.

it is necessary to employ two iterated integrals, because if $0\le x\le 6$, the lower boundary is the graph of $y=0$, whereas if $6\le x\le 9$, the lower boundary is the graph of $y=\sqrt{3x-18}$. Let R_1 denote the part of the region R that lies between $x=0$ and $x=6$, and let R_2 denote the part between $x=6$ and $x=9$ then

$$\begin{aligned}\iint_R f(x,y)dA &= \iint_{R_1} f(x,y)dA + \iint_{R_2} f(x,y)dA\\ &= \int_0^6\int_0^{\sqrt{x}} f(x,y)dydx + \int_6^9\int_{\sqrt{3x-18}}^{\sqrt{x}} f(x,y)dydx.\end{aligned}$$

If we wish to use horizontal slicing then we must solve each of the given equations for x in terms of y, obtaining

$$x = y^2 \quad \text{and} \quad x = \frac{1}{3}y^2 + 6.$$

Only one iterated integral is required in this case and that is

$$\iint_R f(x,y)dA = \int_0^3 \int_{y^2}^{\frac{1}{3}y^2+6} f(x,y)dxdy.$$

Q.4 The region of integration D is the region below the line $x + 3y = 7$ or equivalently $y = (7-x)/3$ and above the line $y = 1$ for $1 \le x \le 4$ (see, Figure 2). Thus,

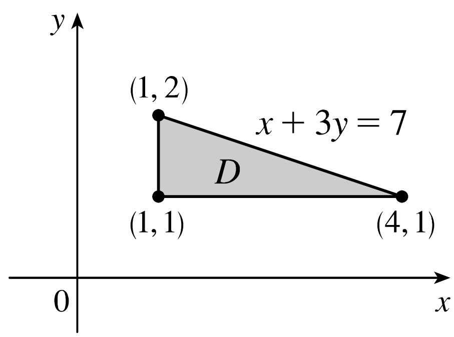

Figure 2: The region of integration in $Q.4$.

$$V = \int_1^4 \int_1^{(7-x)/3} xydydx = \int_1^4 \left[\frac{xy}{2}\right]_1^{(7-x)/3} dx = \int_1^4 \frac{1}{2}x\left[\frac{1}{9}(7-x)^2 - 1\right] dx$$
$$= \frac{1}{18}\int_1^4 \left[x^3 - 14x^2 + 40x\right] dx = \frac{1}{18}\left[\frac{x^4}{4} - \frac{14x^3}{3} + 20x^2\right]_1^4 = \frac{1}{18}\left(\frac{256}{3} - \frac{187}{12}\right) = \frac{31}{8}.$$

Q.5 The region R of the integration is illustrated in Figure 3. The left-hand and right-hand boundaries are the graphs of $x = \sqrt{y}$ and $x = 2$, respectively, and $0 \le y \le 4$. Note that the region R has the lower and upper boundaries at $y = 0$ and $y = x^2$, respectively, and $0 \le x \le 2$. Hence,

$$\int_0^4 \int_{\sqrt{y}}^2 y\cos(x^5)dxdy = \iint_R y\cos(x^5)dA = \int_0^2 \int_0^{x^2} y\cos(x^5)dydx = \int_0^2 \left[\frac{y^2}{2}\cos(x^5)\right]_0^{x^2} dx$$
$$= \int_0^2 \frac{x^4}{2}\cos(x^5)dx = \frac{1}{10}\int_0^2 \cos(x^5)[5x^4dx]$$
$$= \frac{1}{10}\left[\sin(x^5)\right]_0^2 = \frac{1}{10}\sin(32).$$

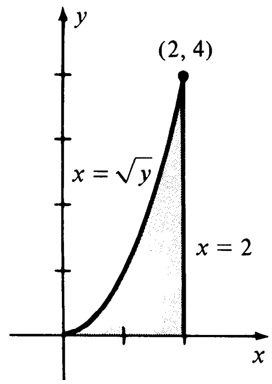

Figure 3: The region of integration in $Q.5$.

Q.6 The region of integration is bounded from above by the line $y = 8$, from below by $x-$axis, from left by the curve $x = y^{1/3}$, and from right by the line $x = 2$ (see, Figure 4).
$$\int_0^8 \int_{y^{1/3}}^2 \sqrt{x^4+1}dxdy = \int_0^2 \left(\int_0^{x^3} \sqrt{x^4+1}dy \right) dx = \int_0^2 x^3\sqrt{x^4+1}dx = \frac{2}{12}(x^4+1)^{3/2}\Big|_0^2.$$

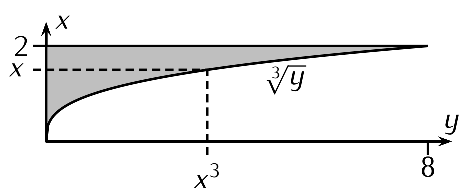

Figure 4: The region of integration in $Q.6$.

Therefore, $\displaystyle\int_0^8 \int_{y^{1/3}}^2 \sqrt{x^4+1}dxdy = \frac{1}{6}\left[(2^4+1)^{3/2} - (0+1)^{3/2}\right] = \frac{1}{6}(17^{3/2}-1).$

“Shine like the whole universe is yours.” — Rumi