Elastic Scattering Coefficients

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Abstract: The notion of elastic scattering coefficients (ESC) is introduced to address a broad range of inverse scattering and imaging problems in elastic media. The link between scattering amplitudes and ESC of small inclusions is established.

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1. Introduction

The notion of *scattering coefficients* for acoustic and electromagnetic inclusions emerged in an effort to design enhanced near invisibility cloaks [4, 5]. These mathematical objects contain rich information of the contrast of material parameters, high order shape oscillations, frequency profile, and the maximum resolving power. They have been effectively used for inverse medium scattering [3], echo-location and shape description [6], and mathematical understanding of super-resolution phenomena in imaging [2]. In electromagnetic or acoustic media, scattering coefficients provide a natural extension to the concept of *contracted polarization tensors* with respect to frequency dependence. They are defined in terms of the Fourier-Bessel coefficients (in 2D) or spherical harmonic coefficients (in 3D) of the far-field scattering amplitude and can be retrieved with high accuracy from the multi-static response data by solving a least-squares optimization problem.

The impetus behind this study is the mathematical imaging of small elastic inclusions of diminishing characteristic size. Our focus is on introducing two dimensional ESC using cylindrical eigen-vectors of the Lamé equation and on establishing their role in elastic scattering. To this end, a few preliminary results are summarized in Section 2. Then, ESC are defined and related to the far field elastic scattering amplitudes in Section 3.

2. Preliminaries and Mathematical Formulation

Let $\mathbb{R}^2$ be loaded with a linear isotropic elastic material possessing homogeneous Lamé parameters $\lambda_0, \mu_0 \in \mathbb{R}_+$ and density $\rho_0 \in \mathbb{R}_+$. Let $D \subset \mathbb{R}^2$ with connected boundary ∂D and homogeneous parameters $\lambda_1, \mu_1, \rho_1 \in \mathbb{R}_+$ be a sufficiently smooth open bounded elastic inclusion such that $(\lambda_0 - \lambda_1)(\mu_0 - \mu_1) > 0$. For any smooth vector field $\mathbf{w}$, let us define the elasticity and surface traction operators $\Delta_a^e[\mathbf{w}]$ and $T_\nu[\mathbf{w}]$ respectively by

$$\Delta_e^a[\mathbf{w}] := \mu_a \Delta \mathbf{w} + (\lambda_a + \mu_a)\nabla\nabla\cdot\mathbf{w} \quad \text{and} \quad T_\nu^a[\mathbf{w}] := \lambda_a(\nabla\cdot\mathbf{w})\nu + \mu_a\left(\nabla\mathbf{w} + (\nabla\mathbf{w})^t\right), \quad a = 0, 1, \tag{1}$$

where ν is the outward unit normal to ∂D and t reflects matrix transpose. Henceforth $c_P = \sqrt{(\lambda_0 + 2\mu_0)/\rho_0}$, $c_S = \sqrt{\mu_0/\rho_0}$, $k_\alpha = \omega/c_\alpha$, $\alpha, \beta \in \{P, S\}$ and $\omega \in \mathbb{R}_+$. Let $\mathbf{G}_\omega^a(x,y)$ be the fundamental solution to the Lamé system $-(\Delta_a^e + \rho_a\omega^2)$ in $\mathbb{R}^2$. Then, we define the single layer potential S_D^a by

$$S_D^a[\psi](x) := \int_{\partial D} \mathbf{G}_\omega^a(x,y)\psi(y)d\sigma(y), \quad x \in \mathbb{R}^2, \quad \psi \in L^2(\partial D)^2. \tag{2}$$

Let $\mathbf{U}$ and $\mathbf{u}$ be respectively the incident field and resulting total field in $\mathbb{R}^2$ satisfying the Lamé equations

$$\Delta_e^0[\mathbf{U}] + \rho_0\omega^2\mathbf{U} = \mathbf{0} \quad \text{and} \quad \begin{cases} \mu\Delta\mathbf{u} + (\lambda+\mu)\nabla\nabla\cdot\mathbf{u} + \rho\omega^2\mathbf{u} = \mathbf{0} \\ (\mathbf{u} - \mathbf{U}) \quad \text{satisfies the Kupradze's radiation conditions,} \end{cases} \tag{3}$$

$$\text{with} \quad \mu = \left(\mu_0\chi_{\mathbb{R}^2\setminus\overline{D}} - \mu_1\chi_D\right), \quad \lambda = \left(\lambda_0\chi_{\mathbb{R}^2\setminus\overline{D}} - \lambda_1\chi_D\right), \quad \rho = \left(\rho_0\chi_{\mathbb{R}^2\setminus\overline{D}} - \rho_1\chi_D\right). \tag{4}$$

Then, the total field $\mathbf{u}$ admits the integral representation

$$\mathbf{u}(x) = \mathbf{U}(x) + S_D^0[\psi](x), \quad x \in \mathbb{R}^2 \setminus \overline{D} \quad \text{and} \quad \mathbf{u}(x) = S_D^1[\varphi](x), \quad x \in D, \tag{5}$$

where the pair $(\varphi, \psi) \in L^2(\partial D)^2 \times L^2(\partial D)^2$ satisfies

$$S_D^1[\varphi] - S_D^0[\psi] = \mathbf{U}|_{\partial D} \quad \text{and} \quad T_\nu^1 S_D^1[\varphi]\big|_- - T_\nu^0 S_D^0[\psi]\big|_= T_\nu^0 \mathbf{U}|_{\partial D} \quad \left(\text{with } w(x)\big|_\pm := \lim_{\varepsilon\to 0^+} w(x \pm \varepsilon\nu)\right). \tag{6}$$

Following result on unique solvability of (6) holds; see, for instance, [1, Theorem 1.7].

Theorem 2.1. *Let D be a Lipschitz domain such that $\omega^2\rho_1$ is different from Dirichlet eigenvalues of the operator $-\Delta_e^1$ on D. Then, for any $\left(\mathbf{U}, T_\nu^0\mathbf{U}\right) \in H^1(\partial D)^2 \times L^2(\partial D)^2$ there exists a unique solution $(\varphi, \psi) \in L^2(\partial D)^2 \times L^2(\partial D)^2$ of (6). Moreover, there exists a constant $C \in \mathbb{R}_+$ such that*

$$\|\varphi\|_{L^2(\partial D)^2} + \|\psi\|_{L^2(\partial D)^2} \leq C\left(\|\mathbf{U}\|_{H^1(\partial D)^2} + \left\|T_\nu^0\mathbf{U}\right\|_{L^2(\partial D)^2}\right). \tag{7}$$

3. Elastic Scattering Coefficients

Let $H_m^{(1)}$ and J_m be the order $m \in \mathbb{Z}$ Hankel and Bessel functions of first kind respectively. For each $k > 0$, $m \in \mathbb{Z}$, let

$$u_m(x,k) := J_m(k|x|)e^{im\varphi_x} \quad \text{and} \quad v_m(x,k) := H_m^{(1)}(k|x|)e^{im\varphi_x}, \quad \text{where} \quad x = (|x|\cos\varphi_x, |x|\sin\varphi_x). \tag{8}$$

Let us also introduce the cylindrical longitudinal and transverse eigen-vectors of the Lamé equation by

$$\mathbf{U}_m^P(x,k_P) := \nabla u_m(x,k_P), \quad \mathbf{U}_m^S(x,k_S) := \nabla \times (\mathbf{e}_3 u_m(x,k_S)) \tag{9}$$

$$\mathbf{V}_m^P(x,k_P) := \nabla v_m(x,k_P), \quad \mathbf{V}_m^S(x,k_S) := \nabla \times (\mathbf{e}_3 v_m(x,k_s)), \tag{10}$$

where $\mathbf{e}_3$ is the unit normal vector to the $(x_1,x_2)-$plane. The following result holds on the completeness and linear independence of $\left(\mathbf{U}_m^P, \mathbf{U}_m^S\right)$ and $\left(\mathbf{V}_m^P, \mathbf{V}_m^S\right)$ with respect to $L^2(\partial D)^2$ norm; see, for instance, [8, Lemmas 1-3].

Lemma 3.1. *Let $D \subset \mathbb{R}^2$ be simply connected domain containing origin and ∂D be a closed Lyapunov curve. Then, the set $\{\mathbf{V}_m^P, \mathbf{V}_m^S : m \in \mathbb{Z}\}$ is complete and linearly independent in $L^2(\partial D)^2$. Moreover, if $\rho_1\omega^2$ is not a Dirichlet eigenvalue of $-\Delta_e^1$ on D, then the set $\{\mathbf{U}_m^P, \mathbf{U}_m^S : m \in \mathbb{Z}\}$ is also complete and linearly independent in $L^2(\partial D)^2$.*

As a consequence of Lemma 3.1, for every incident field $\mathbf{U}$ there exist constants $a_m^P, a_m^S \in \mathbb{C}$ for all $m \in \mathbb{Z}$ such that

$$\mathbf{U}(x,\omega) = \sum_{m\in\mathbb{Z}} \left(a_m^S \mathbf{U}_m^S(x,k_S) + a_m^P \mathbf{U}_m^P(x,k_P)\right). \tag{11}$$

In particular, for any direction of incidence $\mathbf{d} = (\cos\theta, \sin\theta)$ with $\mathbf{d}^\perp \cdot \mathbf{d} = 0$, a general plane wave

$$\mathbf{U}(x,\omega) = \frac{1}{\rho_0 c_S^2} e^{ik_S x\cdot\mathbf{d}}\,\mathbf{d}^\perp + \frac{1}{\rho_0 c_P^2} e^{ik_P x\cdot\mathbf{d}}\,\mathbf{d} = -\left(\frac{i}{\rho_0 c_S^2 k_S}\nabla \times \left[\mathbf{e}_3\, e^{ik_S x\cdot\mathbf{d}}\right] + \frac{i}{\rho_0 c_P^2 k_P}\left[\nabla e^{ik_P x\cdot\mathbf{d}}\right]\right) \tag{12}$$

can be written in the form (11) with $a_m^\beta(\mathbf{U}) = -ie^{im(\pi/2-\theta)}/\rho_0 c_\beta^2 k_\beta$. In fact, this is a simple consequence of Jacobi-Anger decomposition of the plane wave $e^{ikx\cdot\mathbf{d}} = \sum_{m\in\mathbb{Z}} e^{im(\pi/2-\theta)} J_m(k|x|)e^{im\theta_x}$. Note also that for all $x,y \in \mathbb{R}^2$ such that $|x| > |y|$ and for any vector $\mathbf{p} \in \mathbb{R}^2$ (independent of x)

$$\mathbf{G}_\omega^0(x,y)\mathbf{p} = -\frac{i}{4\rho_0 c_S^2}\sum_{n\in\mathbb{Z}} \mathbf{V}_n^S(x,k_S)\left[\overline{\mathbf{U}_n^S(y,k_S)}\cdot\mathbf{p}\right] - \frac{i}{4\rho_0 c_P^2}\sum_{n\in\mathbb{Z}} \mathbf{V}_n^P(x,k_P)\left[\overline{\mathbf{U}_n^P(y,k_P)}\cdot\mathbf{p}\right]. \tag{13}$$

Consequently, thanks to integral representation (5), for all $x \in \mathbb{R}^2 \setminus \overline{D}$

$$\mathbf{u}(x) - \mathbf{U}(x) = -\frac{i}{4\rho_0}\sum_{n\in\mathbb{Z}}\left[\frac{1}{c_P^2}\mathbf{V}_n^S(x,k_S)\int_{\partial D}\left[\overline{\mathbf{U}_n^S(y,k_S)}\cdot\psi(y)\right]d\sigma(y) + \frac{1}{c_S^2}\mathbf{V}_n^P(x,k_P)\int_{\partial D}\left[\overline{\mathbf{U}_n^P(y,k_P)}\cdot\psi(y)\right]d\sigma(y)\right]. \tag{14}$$

We are now fully equipped to define the ESC associated with inclusion D.

Definition 3.2. *Let* $\left(\varphi_m^\beta, \psi_m^\beta\right)$ *be the solution of* (6) *with* $\mathbf{U} = \mathbf{U}_m^\beta$ *for all* $m \in \mathbb{Z}$*. Then, ESC of* D *are defined by*

$$W_{n,m}^{\alpha,\beta} = W_{n,m}^{\alpha,\beta}[D, \lambda_0, \lambda_1, \mu_0, \mu_1, \rho_0, \rho_1, \omega] := \int_{\partial D} \left[\overline{\mathbf{U}_n^\alpha(y, k_\alpha)} \cdot \psi_m^\beta(y)\right] d\sigma(y), \quad n, m \in \mathbb{Z}. \tag{15}$$

The following properties of the ESC can be proved after fairly easy manipulations.

Theorem 3.3. *There exist constants* $C_{\alpha,\beta} > 0$ *such that for each mode* $\alpha, \beta = P, S$

$$\left|W_{m,n}^{\alpha,\beta}[D, \lambda_0, \lambda_1, \mu_0, \mu_1, \rho_0, \rho_1, \omega]\right| \leq C_{\alpha,\beta}^{|n|+|m|-2} |n|^{1-|n|} |m|^{1-|m|}, \quad \forall m, n \in \mathbb{Z}, \ |m|, |n| \to \infty. \tag{16}$$

Moreover, with superposed bar reflecting complex conjugate,

$$W_{m,n}^{\alpha,\beta} = \overline{W_{n,m}^{\beta,\alpha}} = (-1)^{m+n} \overline{W_{-m,-n}^{\alpha,\beta}} \quad \textit{for all} \quad m, n \in \mathbb{Z}. \tag{17}$$

3.1. Connection Between ESC, Scattered Field and Far Field Scattering Amplitudes

Suppose that $\mathbf{U}$ is of the form (12) with decomposition (11). Then, by superposition principle,

$$\psi(x) = \sum_{m \in \mathbb{Z}} \left[a_m^P \psi_m^P + a_m^S \psi_m^S\right] \quad \text{and} \quad \varphi(x) = \sum_{m \in \mathbb{Z}} \left[a_m^P \varphi_m^P + a_m^S \varphi_m^S\right]. \tag{18}$$

This, together with Definition 3.2 and expansion (14), renders the expansion

$$\mathbf{u}(x) - \mathbf{U}(x) = \sum_{n,m \in \mathbb{Z}} \left[\left(\gamma_m^P W_{m,n}^{P,P} + \gamma_m^S W_{m,n}^{P,S}\right) \mathbf{V}_n^P(x, k_P) + \left(\gamma_m^P W_{m,n}^{S,P} + \gamma_m^S W_{m,n}^{S,S}\right) \mathbf{V}_n^S(x, k_S)\right] \quad \text{as } |x| \to \infty, \tag{19}$$

where $\gamma_m^\alpha := -i a_m^\alpha / 4\rho_0 c_\alpha^2$. Moreover, thanks to far field behavior of $H_n^{(1)}$ [7, Formulae 10.2.5 and 10.17.11],

$$\mathbf{V}_n^P(x, k_P) \sim \frac{e^{ik_P|x|}}{\sqrt{|x|}} A_n^{\infty,P} e^{in\phi_x} \hat{\mathbf{e}}_r \quad \text{and} \quad \mathbf{V}_n^S(x, k_S) \sim \frac{e^{ik_S|x|}}{\sqrt{|x|}} A_n^{\infty,S} e^{in\phi_x} \hat{\mathbf{e}}_\theta \quad \text{as } |x| \to \infty, \tag{20}$$

where $\hat{\mathbf{e}}_r$ and $\hat{\mathbf{e}}_\theta$ are the radial and angular unit vectors and

$$A_n^{\infty,P} := (i+1) k_P e^{-in\pi/2} \sqrt{k_P/\pi} \quad \text{and} \quad A_n^{\infty,S} := -(i+1) k_S e^{-in\pi/2} \sqrt{k_S/\pi}. \tag{21}$$

Thus, the following result is proved.

Theorem 3.4. *For incident field* $\mathbf{U}$ *given by* (11)*, longitudinal and transverse scattering amplitudes are given by*

$$\mathbf{u}_P^\infty[D](\hat{x}) = \sum_{n,m \in \mathbb{Z}} \left(\gamma_m^P W_{m,n}^{P,P} + \gamma_m^S W_{m,n}^{P,S}\right) A_n^{\infty,P} e^{in\varphi_x} \hat{\mathbf{e}}_r \quad \textit{and} \quad \mathbf{u}_S^\infty[D](\hat{x}) = \sum_{n,m \in \mathbb{Z}} \left(\gamma_m^P W_{m,n}^{S,P} + \gamma_m^S W_{m,n}^{S,S}\right) A_n^{\infty,S} e^{in\varphi_x} \hat{\mathbf{e}}_\theta. \tag{22}$$

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