National University of Technology, Islamabad
End Term Exam (Calculus II), Spring 2019
Solution Key

Q.1 Suppose that $x = 2u + v$, $y = u/v$ and $z = e^{xy}$. Use an appropriate form of the chain rule to find $\partial z/\partial u$ and $\partial z/\partial v$.

Sol. We have

$$\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial y}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = \left(ye^{xy}\right)(2) + \left(xe^{xy}\right)\left(\frac{1}{v}\right) = \left(2y + \frac{x}{v}\right)e^{xy} \\ &= \left(\frac{2u}{v} + \frac{2u+v}{v}\right)e^{(2u+v)(u/v)} = \left(\frac{4u}{v} + 1\right)e^{(2u+v)(u/v)},\end{aligned}$$

and

$$\begin{aligned}\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = \left(ye^{xy}\right)(1) + \left(xe^{xy}\right)\left(-\frac{u}{v^2}\right) = \left(y - \frac{xu}{v^2}\right)e^{xy} \\ &= \left(\frac{u}{v} - \frac{(2u+v)u}{v^2}\right)e^{(2u+v)(u/v)} = -\left(\frac{2u^2}{v^2}\right)e^{(2u+v)(u/v)}.\end{aligned}$$

Q.2 Locate all the relative extrema and saddle points of $f(x, y) = 3x^2 - 2xy + y^2 - 8y$.

Sol. Since $f_x(x, y) = 6x - 2y$ and $f_y(x, y) = -2x + 2y - 8$, the critical points of f satisfy the equations

$$f_x(x, y) = 6x - 2y = 0, \qquad f_y(x, y) = -2x + 2y - 8 = 0.$$

Solving these equations for x and y yields $x = 2$ and $y = 6$, so $(2, 6)$ is the only critical point.

We also need the second-order partial derivatives in order to apply the second derivative test for extrema and saddle points. We have

$$f_{xx}(x, y) = 6, \qquad f_{yy}(x, y) = 2, \qquad f_{xy}(x, y) = -2.$$

At the point $(2, 6)$, we have

$$D = f_{xx}(2, 6)f_{yy}(2, 6) - f_{xy}^2(2, 6) = 6(2) - (-2)^2 = 8 > 0,$$

and

$$f_{xx}(2, 6) = 6 > 0.$$

So, f has a relative minimum at $(2, 6)$ by second derivative test.

Q.3 Suppose that the temperature at a point (x, y) on a metal plate is $T(x, y) = 4x^2 - 4xy + y^2$. An ant, walking on the plate, traverses a circle of radius 5 centered at the origin. What are the highest and lowest temperatures encountered by the ant?

Sol. The ant is constrained to traverse in a circle of radius 5. Therefore, assuming the center of the circle to be the origin, the constraint equation is $x^2 + y^2 = 25$. Thus, the objective function is $T(x, y) = 4x^2 - 4xy$ and the constraint function is $g(x, y) = x^2 + y^2 - 25$. By using method of Lagrange multipliers, we set $\nabla T = \lambda \nabla g$, i.e.,

$$(8x - 4y)\mathbf{i} - 4x\mathbf{j} = \lambda(2x\mathbf{i} + 2y\mathbf{j}).$$

In components, we have $8x - 4y = 2x\lambda$ and $-4x + 2y = 2y\lambda$. Note that x and y cannot be zero, because if $x = 0$ then $y = 0$ and conversely; however, $x^2 + y^2 = 25$. Therefore, both x and y are non-zero. Thus, $\lambda = (4x - 2y)/x$ and $\lambda = (-2x + y)/y$ from the component equations above. So

$$(4x - 2y)/x = (-2x + y)/y \iff 2x^2 + 3xy - 2y^2 = 0 \iff (2x - y)(x + 2y) = 0.$$

Therefore, either $y = 2x$ or $x = -2y$. If $y = 2x$ then $x^2 + (2x)^2 = 25$, i.e., $x = \pm\sqrt{5}$. If $x = -2y$ then $(-2y)^2 + y^2 = 25$, i.e., $y = \pm\sqrt{5}$. Therefore, the possible points of extrema are $(-\sqrt{5}, -2\sqrt{5})$, $(\sqrt{5}, 2\sqrt{5})$, $(-2\sqrt{5}, \sqrt{5})$ and $(2\sqrt{5}, -\sqrt{5})$. Since

$$T(-\sqrt{5}, -2\sqrt{5}) = 0 = T(\sqrt{5}, 2\sqrt{5}) \quad \text{and} \quad T(-2\sqrt{5}, \sqrt{5}) = 125 = T(2\sqrt{5}, -\sqrt{5}).$$

The highest temperature is 125 and the lowest temperature is 0 subject to the constraint $x^2 + y^2 = 25$.

Q.4 The rooftop of a building is designed in the form of an inclined plane through the point $(4, 3, 0)$ and parallel to the beams represented by the vectors $\mathbf{i} + \mathbf{k}$ and $2\mathbf{j} - \mathbf{k}$. Find the equation of the rooftop.

Sol. Since the plane is parallel to the beam vectors $\mathbf{i} + \mathbf{k}$ and $2\mathbf{j} - \mathbf{k}$. The normal to the plane is also normal to the beams. Therefore, a normal vector (denoted $\mathbf{n}$) on the rooftop is given by

$$\mathbf{n} = (\mathbf{i} + \mathbf{k}) \times (2\mathbf{j} - \mathbf{k}) = \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{pmatrix} = -2\mathbf{i} + \mathbf{j} + 2\mathbf{k}.$$

Let $P(x, y, z)$ be an arbitrary point on the plane representing the rooftop passing through the point $P_0(4, 3, 0)$. Then, the equation of the rooftop is given by

$$\mathbf{n} \cdot \vec{P_0P} = 0 \implies -2(x - 4) + 1(y - 3) + 2(z - 0) = 0 \implies -2x + y + 2z = -5.$$

This is the required equations.

Q.5 Find the triple integral of $f(x, y, z) = z$ over the slice of the hemisphere shown in Figure 1 using the triangular "shadow" in the xy-plane.

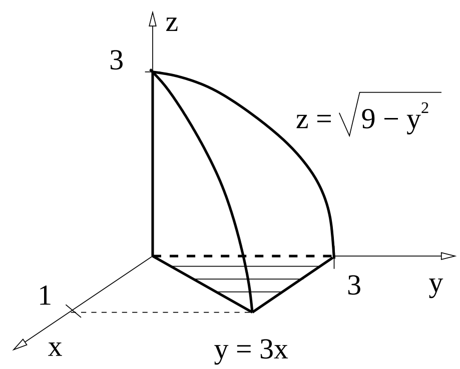

Figure 1: Hemisphere and the shadow region.

Sol. Note that the slice of the hemisphere in Figure 1 is bounded by the planes $y = 3x$, $x = 0$ and $z = 0$ and by the surface $z = \sqrt{9 - y^2}$. Moreover the shadow region is defined by the triangle in the xy-plane bounded by the lines $y = 3x$, $y = 3$ and $x = 0$. Therefore, by method of slicing a vertical line parallel to z-axis will enter the hemispherical domain from $z = 0$ and leave it from the surface $z = \sqrt{9 - y^2}$. If we slice the shadow by lines parallel to y-axis, then such lines will render the x- and y-limits as $x \in [0, 1]$ and $y \in [3x, 3]$. (We could also have chosen to slice the shadow region by lines parallel to x-axis, then the x- and y- limits will be be $y \in [0, 3]$ and $x \in [0, y/3]$).

Therefore, the required triple integral of $f(x, y, z) = z$ over the hemisphere is

$$\begin{aligned}\iiint_D f(x,y,z)dV &= \int_0^1 \int_{3x}^{x} \int_0^{\sqrt{9-y^2}} z dz dy dx = \int_0^1 \int_{3x}^{3} \left(\frac{z^2}{2} \Big|_0^{\sqrt{9-y^2}} \right) dy dx \\ &= \frac{1}{2} \int_0^1 \int_{3x}^{3} \left(9 - y^2\right) dy dx = \frac{1}{2} \int_0^1 \left[9y - \frac{y^3}{3} \right] \Big|_{3x}^{3} dx \\ &= \frac{9}{2} \int_0^1 \left[3(1-x) - (1-x)^3 \right] dx = -\frac{9}{2} \left[\frac{3}{2}(1-x)^2 - \frac{1}{4}(1-x)^4 \right] \Big|_0^1 \\ &= \frac{9}{2} \left(\frac{3}{2} - \frac{1}{4} \right) = \frac{45}{8}.\end{aligned}$$

Q.6 Express the integral $I = \int_0^2 \int_0^y x dx dy$ as a polar integral.

Sol. First we sketch the integration region in Figure 2. The outer integration limit is $y \in [0, 2]$

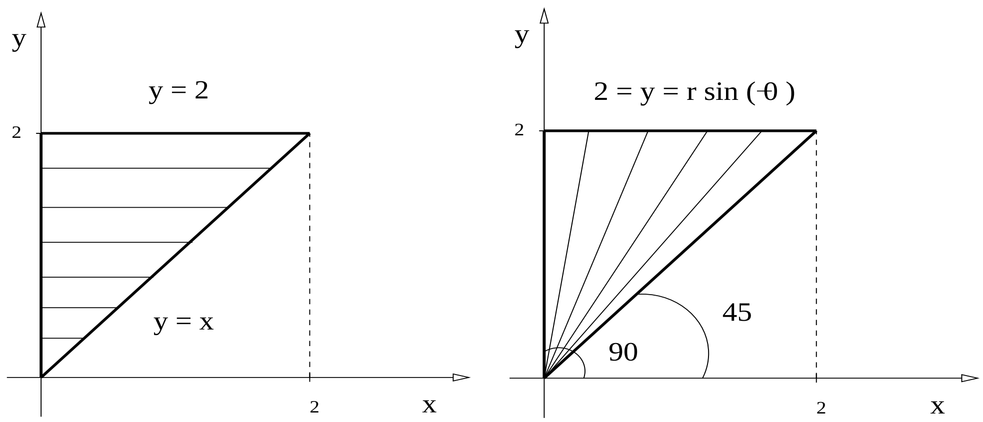

Figure 2: Region of intergation in cartesian (left) and polar coordinates (right).

and the for every y, the x coordinate satisfies $x \in [0, y]$. The upper limit for x is the curve $y = x$. It is simple to describe this domain in polar coordinates since $y = x$ is $\theta_1 = \frac{\pi}{4}$, the line $x = 0$ is $\theta_2 = \frac{\pi}{2}$. From the right figure in 2, one can easily see that the lower integration limit in r is $r = 0$ and the upper limit is $2 = y = r \sin\theta$ or $r = 2/\sin\theta$. Finally, recall that $x = r\cos\theta$ and $y = r\sin\theta$. Therefore, we conclude that

$$\int_0^2 \int_0^y x dx dy = \int_{\pi/4}^{\pi/2} \int_0^{2/\sin\theta} (r\cos\theta) r dr d\theta.$$

Q.7 Evaluate the integral $\displaystyle\int_0^3 \int_{x^2}^9 x^3 e^{y^3} dy dx$ by first reversing the order of integration.

Sol. If we tried to integrate with respect to y first, we cannot do it. Notice first that the region of double integration has two properties: $0 \leq x \leq 3$ and $x^2 \leq y \leq 9$. We then can draw the region (see, Figure 3).

By the method of slicing, we invade the region by arbitrary horizontal line that gives us the y-limits $0 \leq y \leq 9$. One the other hand an arbitrary horizontal line meets the region at two points $x = 0$ and when $x^2 = y$ or simply $x = \sqrt{y}$ (we consider the positive root because we are handling the region in the first quadrant). Therefore, the x- limits are $x = 0$ and $x = \sqrt{y}$.

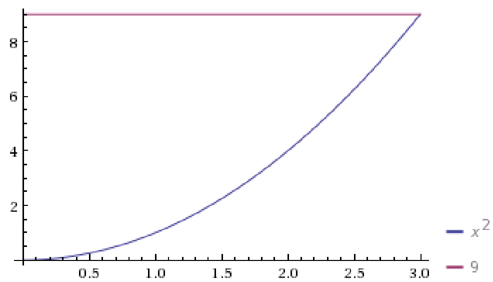

Figure 3: Region of intergation Q.7.

Thus, the required integral in the reverse order is given by

$$\int_0^3 \int_{x^2}^9 x^3 e^{y^3} dydx = \int_0^9 \int_0^{\sqrt{y}} x^3 e^{y^3} dxdy = \int_0^9 \left(\frac{1}{4}x^4 e^{y^3}\right)\Big|_0^{\sqrt{y}} dy = \frac{1}{4}\int_0^9 y^2 e^{y^3} dy$$
$$= \frac{1}{12} e^{y^3}\Big|_0^9 = \frac{1}{12}\left(e^{729} - 1\right).$$

Q.8 Suppose that a semicircular wire has the equation $y = \sqrt{25 - x^2}$ and that its mass density is $\delta(x, y) = 15 - y$. Find the mass of the wire using line integrals over the curve C representing the wire and the standard parametrization of the semi-circle, i.e., $x(\theta) = r\cos\theta$ and $y(\theta) = r\sin\theta$ with parameter θ.

Sol. The mass M of the wire can be expressed as the line integral

$$M = \int_C \delta(x, y) ds = \int_C (15 - y) ds$$

along the semicircle C. To evaluate this integral, we will express C parametrically as

$$x = 5\cos\theta, \qquad y = 5\sin\theta, \qquad (0 \le \theta \le \pi).$$

Note also that the displacement and consequently the velocity vector of the point moving along the curve C are given by $\mathbf{r}(\theta) := 5\cos\theta\mathbf{i} + 5\sin\theta\mathbf{j}$, $\mathbf{v}(\theta) := -5\sin\theta\mathbf{i} + 5\cos\theta\mathbf{j}$. Consequently, $|\mathbf{v}(\theta)| = \sqrt{(-5\sin\theta)^2 + (5\cos\theta)^2} = \sqrt{25(\sin\theta^2 + \cos\theta^2)} = 5$. Therefore, we have

$$M = \int_C (15 - y) ds = \int_0^\pi (15 - 5\sin\theta)|v(\theta)| d\theta = \int_0^\pi (15 - 5\sin\theta) 5 d\theta = 25\Big[3\theta - \sin\theta\Big]\Big|_0^\pi = 75\pi - 50.$$

"If you believe it will work out, you'll see opportunities. If you believe it won't, you will see obstacles." — Wayne Dyer