SCHOOL OF NATURAL SCIENCES
DEPARTMENT OF MATHEMATICS
FINAL EXAM, (MATH-455 INTEGRAL EQUATIONS)
SOLUTION KEY

Q1. Consider the Volterra integral equations

$$\phi(s) = s^2 + \int_0^s (s-t)\phi(t)dt, \qquad s \in [0,1]. \tag{1}$$

(a) Find an exact solution to equation (1).

Ans. There are at least two possible methods to solve (1) for an exact solution: (1) Laplace transform and (2) by converting the equation to an equivalent initial value problem. We reduce (1) to an initial value problem here.

Note that if ϕ is a continuous solution of (1), then the equation implies that it is also differentiable, hence one may differentiate (1) and get

$$\phi'(s) = 2s + \Big[(s-t)\phi(t)\Big]_{t=s} + \int_0^s \frac{\partial(s-t)}{\partial s}\phi(t)dt = 2s + \int_0^s \phi(t)dt. \tag{a}$$

Differentiating this once more one finally gets a simple ordinary differential equation

$$\phi''(s) = 2 + \phi(s). \tag{b}$$

In order that this is uniquely solvable, one also needs suitable initial conditions. These follow immediately from the integral equation itself. First of all, evaluating (1) at $s = 0$ one gets

$$\phi(0) = 0, \tag{c}$$

and from the once differentiated equation (a) one obtains

$$\phi'(0) = 2 + \phi(0) = 2. \tag{d}$$

It is easy to see that the unique solution of the equation (b) is

$$\phi(s) = e^s + e^{-1} - 2. \tag{e}$$

Indeed, $\phi_c(s) = C_1 e^s + C_2 e^{-s}$ and $\phi_p = -2$, with $C_1 = 1$ and $C_2 = 1$ thanks to conditions (c) and (d).

(b) Find Neumann series solution of (1) using the method of successive approximations and verify with the exact solution.

Ans. Note that $\lambda = 1$, $f(s) = s^2$, and $K(x,t) = s - t$ in (1). The functions f and K are continuous and bounded on $[0,1]$. Suppose the initial guess

$$\phi^{(0)}(s) = s^2 = \frac{2(s^2)^1}{(2\cdot 1)!}.$$

Then,

$$
\begin{aligned}
\phi^{(1)}(s) =& \int_0^s K(s,t)\phi^{(0)}(t)dt = \int_0^s (s-t)t^2 dt \\
=& \frac{s^4}{3} - \frac{s^4}{4} = \frac{s^4}{12} = \frac{2(s^2)^2}{4!} = \frac{2(s^2)^2}{(2\cdot 2)!}, \\
\phi^{(2)}(s) =& \int_0^s K(s,t)\phi^{(1)}(t)dt = \int_0^s (s-t)\frac{t^4}{12} dt \\
=& \frac{s^6}{60} - \frac{s^6}{72} = \frac{s^6}{360} = \frac{2(s^2)^3}{6!} = \frac{2(s^2)^3}{(2\cdot 3)!}, \\
\phi^{(3)}(s) =& \int_0^s K(s,t)\phi^{(2)}(t)dt = \int_0^s (s-t)\frac{t^6}{360} dt \\
=& \frac{s^8}{2520} - \frac{s^8}{2880} = \frac{s^8}{20160} = \frac{2(s^2)^4}{4!} = \frac{2(s^2)^4}{(2\cdot 4)!}, \\
& \vdots
\end{aligned}
$$

Thus, the approximate solutions are given by

$$
\begin{aligned}
\phi_0(s) =& \phi^{(0)}(s) = \frac{2(s^2)^1}{(2\cdot 1)!} \\
\phi_1(s) =& \phi^{(0)}(s) + \phi^{(1)}(s) = \frac{2(s^2)^1}{(2\cdot 1)!} + \frac{2(s^2)^2}{(2\cdot 2)!}, \\
\phi_2(s) =& \phi^{(0)}(s) + \phi^{(1)}(s) + \phi^{(2)}(s) = \frac{2(s^2)^1}{(2\cdot 1)!} + \frac{2(s^2)^2}{(2\cdot 2)!} + \frac{2(s^2)^3}{(2\cdot 3)!}, \\
\phi_3(s) =& \phi^{(0)}(s) + \phi^{(1)}(s) + \phi^{(2)}(s) + \phi^{(3)}(s) = \frac{2(s^2)^1}{(2\cdot 1)!} + \frac{2(s^2)^2}{(2\cdot 2)!} + \frac{2(s^2)^3}{(2\cdot 3)!} + \frac{2(s^2)^4}{(2\cdot 4)!}, \\
& \vdots \\
\phi_N(s) =& \sum_{k=0}^{N} \phi^{(k)}(s) = \sum_{k=1}^{N+1} \frac{2s^{2k}}{(2k)!}.
\end{aligned}
\tag{f}
$$

From above, the Neumann series solution of (1) is

$$
\phi(s) = \sum_{k=1}^{\infty} \frac{2s^{2k}}{(2k)!}, \tag{g}
$$

provided the series converges. On the other hand, using the power series expansion of the exponential function, one has

$$
e^s + s^{-s} - 2 = \sum_{k=0}^{\infty} \frac{s^n}{n!} + \sum_{k=0}^{\infty} \frac{(-s)^n}{n!} - 2 = \frac{2s^2}{2!} + \frac{2s^4}{4!} + \frac{2s^6}{6!} + \cdots = \sum_{k=1}^{\infty} \frac{2s^{2k}}{(2k)!},
$$

which is exactly the series solution in (g). Thus, the method of successive approximations produces the beginning of the Taylor-series of the solution.

(c) Discuss the convergence of the series solution.

Ans. We know that the Neumann series solution to a Volterra integral equation of second kind with continuous kernel and source term converges absolutely and uniformly for all values of λ and is continuous. Therefore, the solution given in (g) converges absolutely and uniformly with $\lambda = 1$, $K(s,t) = s - t$, and $f = s^2$.

(d) The kernel operator of the integral equation (1) is square integrable on $[0, 1]$. What effects on the solution do you anticipate had it not been L^2 or a bounded linear operator in the Hilbert space? Explain briefly.

Ans. If the kernel operator of the integral equation is not L^2 or bounded linear operator in a Hilbert space, we could not guarantee the existence of a unique square integrable solution or a bounded solution ϕ. Moreover, we should not expect for sure that the eigenvalues of the homogeneous equation form a discrete set σ (spectrum). We should not also expect that $\Lambda = \sigma^c$ form the set of regular values of the kernel, i.e., we should expect some values of $\lambda \notin \sigma$ corresponding to which there does not exist a solution to the integral equation (1).

Q2. Consider the integral equation

$$\int_0^1 K(x,y)u(y) = 1, \qquad K(x,y) := \begin{cases} xy + (x-y)^{-1/2}, & x > y, \\ xy, & x < y. \end{cases} \tag{2}$$

(a) Convert (2) into an Abel's equation treating

$$\beta := \int_0^1 yu(y)dy, \tag{3}$$

as a known constant initially.

Ans. Note that (2) can be expressed explicitly as

$$\begin{aligned}
1 =& \int_0^x K(x,y)u(y)dy + \int_x^1 K(x,y)u(y)dy \\
=& \int_0^x \left(xy + (x-y)^{-1/2}\right) u(y)dy + \int_x^1 xyu(y)dy \\
=& \int_0^x xyu(y)dy + \int_x^1 xyu(y)dy + \int_0^x (x-y)^{-1/2}u(y)dy \\
=& x\int_0^1 yu(y)dy + \int_0^x (x-y)^{-1/2}u(y)dy \\
=& \beta x + \int_0^x \frac{u(y)}{\sqrt{x-y}}dy.
\end{aligned}$$

Therefore, equation (2) together with β defined in (4) known is equivalent to the Abel's equation

$$\int_0^x K(x,y)u(y)dy = f(x), \quad \text{where} \quad K(x,y) = K(x-y) := \frac{1}{\sqrt{x-y}}, \quad f(x) := 1 - \beta x. \tag{h}$$

(b) Solve the resulting equation using Laplace transform.

Ans. To solve equation (h), we apply Laplace transform to get

$$\hat{K}(s) \cdot \hat{U}(s) = \hat{F}(s) \quad \text{or equivalently} \quad u(x) = \mathcal{L}^{-1}\left[\frac{\hat{F}(s)}{\hat{K}(s)}\right],$$

where $\hat{U}$ is the Laplace transform of u and

$$\hat{K}(s) = \int_0^\infty e^{-sx}x^{-1/2}dx = \mathcal{L}\left[\frac{1}{\sqrt{x}}\right] = \sqrt{\frac{\pi}{s}}, \tag{i}$$

$$\hat{F}(s) = \int_0^\infty e^{-sx}(1-\beta x)dx = \mathcal{L}[1-\beta x] = \frac{1}{s} - \frac{\beta}{s^2}. \tag{j}$$

Therefore,

$$\begin{aligned} u(x) =& \mathcal{L}^{-1}\left[\frac{\hat{F}(s)}{\hat{K}(s)}\right] = \mathcal{L}^{-1}\left[\frac{1}{\sqrt{\pi s}}\left(1-\frac{\beta}{s}\right)\right] = \frac{1}{\pi}\left(\mathcal{L}^{-1}\left[\sqrt{\frac{\pi}{s}}\right] - \beta\mathcal{L}^{-1}\left[\sqrt{\frac{\pi}{s}}\frac{1}{s}\right]\right) \\ =& \frac{1}{\pi}\left(\frac{1}{\sqrt{x}} - \beta\mathcal{L}^{-1}\left[\mathcal{L}\left[\frac{1}{\sqrt{x}}\right]\mathcal{L}[1]\right]\right) = \frac{1}{\pi}\left(\frac{1}{\sqrt{x}} - \beta\frac{1}{\sqrt{x}} * 1\right) \\ =& \frac{1}{\pi}\left(\frac{1}{\sqrt{x}} - \beta\int_0^x \frac{1}{\sqrt{t}}dt\right) = \frac{1}{\pi}\left(\frac{1}{\sqrt{x}} - 2\beta\sqrt{x}\right), \end{aligned}$$

and so we have

$$u(x) = \frac{1}{\pi\sqrt{x}}(1-2\beta x). \tag{k}$$

(c) Use the expression of the solution obtained in the previous part to evaluate constant β and finally furnish the solution to (2).

Ans. Using the expression (k) for $u(x)$ in expression (3) for β to get

$$\begin{aligned} \beta =& \int_0^1 yu(y)dy = \frac{1}{\pi}\int_0^1 y \cdot \frac{1}{\sqrt{y}} \cdot (1-2\beta y)dy = \frac{1}{\pi}\int_0^1 \left(y^{1/2} - 2\beta y^{3/2}\right)dy \\ =& \frac{1}{\pi}\left(\left[\frac{2}{3}y^{3/2}\right]\Big|_0^1 - 2\beta\left[\frac{2}{5}t^{5/2}\right]\Big|_0^1\right) = \frac{1}{\pi}\left(\frac{2}{3} - \frac{4\beta}{5}\right). \end{aligned}$$

On simple manipulations, one gets

$$\beta + \frac{4}{5\pi}\beta = \frac{2}{3\pi} \implies \beta = \frac{2/(3\pi)}{1+4/(5\pi)} = \frac{10}{15\pi+12}. \tag{l}$$

Substituting (l) in (k) to get

$$u(x) = \frac{1}{\pi}\frac{1}{\sqrt{x}}\left(1 - 2\frac{10}{15\pi+12}x\right) = \frac{1}{\pi\sqrt{x}}\left(1 - \frac{20}{15\pi+12}x\right).$$

(d) Can we convert (2) to an integral equation of the second kind? Explain briefly.

Ans. No, we can't convert (2) to an integral equation of the second kind because we require to differentiate (2) with respect to x. However, the kernel to (2) is singular and non-differentiable within the domain of $u(x)$.

Q3. Consider the integral equation

$$\psi(x) = f(x) + \lambda \int_x^\infty e^{a(x-y)}\psi(y)dy, \qquad a > 0,\ x \in \mathbb{R}. \tag{4}$$

(a) Is the kernel of equation (4) square-integrable? Justify your response mathematically.

Ans. The kernel of the equation (4), i.e., $K(x, y) = e^{a(x-y)}$ is not square-integrable. In fact,

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}} |K(x,y)|^2 dxdy &= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{2a(x-y)}dxdy = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{2ax}e^{-2ay}dxdy \\ &= \int_{\mathbb{R}} e^{2ax}dx \int_{\mathbb{R}} e^{-2ay}dy = +\infty\end{aligned}$$

(b) Convert the homogeneous equation associated to (4) (i.e., $f \equiv 0$) to a differential equation and determine the value(s) of λ for which the resulting ODE has non-trivial solution(s) if any.

Ans. The associated homogeneous equation to (4) is

$$\psi(x) = \lambda \int_x^\infty e^{a(x-y)}\psi(y)dy. \tag{m}$$

Differentiating with respect to x on both sides, (m) gives

$$\psi'(x) = -\lambda\psi(x) + \lambda \int_x^\infty ae^{a(x-y)}\psi(y)dy = -\lambda\psi(x) + a\psi(x).$$

Therefore, we get the equation

$$\psi'(x) + (\lambda - a)\psi(x) = 0, \tag{n}$$

which is a separable ordinary differential equation with solution

$$\psi(x) = ce^{-(\lambda-a)x}. \tag{o}$$

Thus, equation (n) has non-trivial solutions for all values of λ.

(c) A requirement on the solution to equation (4) and the associated homogeneous equation is that the integral on the RHS should converge, i.e.,

$$\int_x^\infty e^{a(x-y)}\psi(y) < \infty. \tag{5}$$

Use this requirement and the solution obtained in the previous part to determine the spectrum of (4). Is the spectrum discrete or continuous?

Ans. Let us check the convergence of the integral for ψ given in (o)

$$\begin{aligned}\int_x^\infty e^{a(x-y)}\psi(y)dy =& c\int_x^\infty e^{a(x-y)}e^{-(\lambda-a)y}dy\\ =& ce^{ax}\int_x^\infty e^{-\lambda y}dy\\ =& -ce^{ax}\left[\frac{e^{-\lambda y}}{\lambda}\right]\Big|_x^\infty\\ =& ce^{ax}\left[\frac{e^{-\lambda x}}{\lambda}\right]\\ <& \infty \iff \lambda > 0.\end{aligned}$$

Hence, $\sigma = \Big\{\lambda :\ \lambda > 0\Big\}$ is the spectrum which is continuous.

"Many of life's failures are people who did not realise how close they were to success when they gave up." — Thomas Edison.