School of Natural Sciences
Department of Mathematics
Final Exam, (MATH-455 Integral Equations)
Due Date: June 29, 2020, Time: 3hrs, Marks: 45

Q1. Consider the Volterra integral equations

$$\phi(s) = s^2 + \int_0^s (s-t)\phi(t)dt, \qquad s \in [0,1]. \tag{1}$$

(a) Find an exact solution to equation (1). **[5 Marks]**

(b) Find Neumann series solution of (1) using the method of successive approximations and verify with the exact solution. **[6 Marks]**

(c) Discuss the convergence of the series solution. **[2 Marks]**

(d) The kernel operator of the integral equation (1) is square integrable on $[0,1]$. What effects on the solution do you anticipate had it not been L^2 or a bounded linear operator in the Hilbert space? Explain briefly. **[2 Marks]**

Q2. Consider the integral equation

$$\int_0^1 K(x,y)u(y) = 1, \qquad K(x,y) := \begin{cases} xy + (x-y)^{-1/2}, & x > y, \\ xy, & x < y. \end{cases} \tag{2}$$

(a) Convert (2) into an Abel's equation treating

$$\beta := \int_0^1 yu(y)dy, \tag{3}$$

as a known constant initially. **[4 Marks]**

(b) Solve the resulting equation using Laplace transform (*Hint: Do not follow the procedure in the reference book. Instead, make use of the Laplace transform and invoke the formula* $\mathcal{L}[1/\sqrt{x}] = \sqrt{\pi/s}$). **[5 Marks]**

(c) Use the expression of the solution obtained in the previous part to evaluate constant β and finally furnish the solution to (2). **[4 Marks]**

(d) Can we convert (2) to an integral equation of the second kind? Explain briefly. **[2 Marks]**

Q3. Consider the integral equation

$$\psi(x) = f(x) + \lambda \int_x^\infty e^{a(x-y)}\psi(y)dy, \qquad a > 0, \; x \in \mathbb{R}. \tag{4}$$

(a) Is the kernel of equation (4) square-integrable? Justify your response mathematically. **[5 Marks]**

(b) Convert the homogeneous equation associated to (4) (i.e., $f \equiv 0$) to a differential equation and determine the value(s) of λ for which the resulting ODE has non-trivial solution(s) if any. [**5 Marks**]

(c) A requirement on the solution to equation (4) and the associated homogeneous equation is that the integral on the RHS should converge, i.e.,

$$\int_x^\infty e^{a(x-y)}\psi(y) < \infty. \tag{5}$$

Use this requirement and the solution obtained in the previous part to determine the spectrum of (4). Is the spectrum discrete or continuous? [**5 Marks**]

"Many of life's failures are people who did not realise how close they were to success when they gave up." — Thomas Edison.