National University of Technology, Islamabad
Midterm (Calculus II), Spring 2019
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Solution-Key

Q.1 Let the storage dome of the raw-material (mainly lime stone) at the quarry site of Fouji Cement Company be modeled by the function $f(x,y) = \sqrt{4-x^2-y^2}$. Calculate the domain of f that represents the region in Cartesian coordinates on which the storage dome is constructed. **Explain** the shape of the contour of the dome on the ground (i.e., the contour curve of f when $c = 0$)?
[SLO 1, PLO 1, Marks 3]

Sol. The expression inside the radical must be non-negative, so the domain consists of all ordered pairs satisfying $4 - x^2 - y^2 \geq 0$. Therefore,

$$\text{Domain} = \left\{(x,y) \mid \quad x^2 + y^2 \leq 4\right\}.$$

For the level set, we fix $f(x,y) = c = 0$. This yields

$$\sqrt{4-x^2-y^2} = 0 \implies 4 - x^2 - y^2 = 0 \implies x^2 + y^2 = 4.$$

The shape of the contour of the storage dome is a circle of radius 2units in $xy-$plane.

Q.2 A mason wants to decorate a small room with two different styles of marbles. The total cost of the decoration is modeled as $C = C(x,y) = x^2y - y^2$ where x, y are costs of individual marbles used in the process. At the same time, the prices of both the marbles fluctuate with time as $x = x(t) = e^t - \sinh t$ and $y = y(t) = \cosh t$. **Calculate** the total rate of change of the cost of decoration with time. Express your answer as a function of time.
[SLO 3, PLO 1, Marks 7]

Sol. Using chain rule, we get

$$\frac{dC}{dt} = \frac{\partial C}{\partial x}\frac{dx}{dt} + \frac{\partial C}{\partial y}\frac{dy}{dt}.$$

It is easy to calculate that

$$\frac{\partial C}{\partial x} = 2xy, \quad \frac{\partial C}{\partial y} = x^2 - 2y, \frac{dx}{dt} = e^t - \cosh t, \quad \frac{dy}{dt} = \sinh t.$$

Therefore,

$$\begin{aligned}\frac{dC}{dt} =& 2xy(e^t - \cosh t) + (x^2 - 2y)\sinh t,\\ =& 2(e^t - \sinh t)\cosh t(e^t - \cosh t) + [(e^t - \sinh t)^2 - 2\sinh t]\sinh t.\end{aligned}$$

Q.3 A flat table, modeled with a plane having equation $2x + 2y - z = 3$ is cut by a blade having equation $3x + 6z = 1$. There is nail in the table at point $(1,1,1)$. How far is the nail from the cutting line generated by the blade on the plane? (*Hint: **Calculate** the shortest distance between the nail and the line of intersection of plane and the blade*).
[SLO2, PLO 1, Marks 7]

Sol. We need to find the equation of the line of intersection of the planes $2x + 2y - z = 3$ and $3x + 6z = 1$. Note that the corresponding normals are $\mathbf{n}_b = 3\mathbf{i} + 6\mathbf{k}$ and $\mathbf{n}_t = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$. Therefore, the line of intersection is parallel to

$$\mathbf{n}_t \times \mathbf{n}_b = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & 6 \\ 2 & 2 & -1 \end{pmatrix} = -12\mathbf{i} + 15\mathbf{j} + 6\mathbf{k}.$$

Now, we need to find a point on the line of intersection. For that, we solve the equations of the planes simultaneously. We multiply the equation of the table by 6 and add to the equation of the blade to get

$$15x + 12y = 19.$$

As there are infinite many points on the line, we choose only one by fixing $x = 0$ and then finding $y = 19/12$ and ultimately $z = 1/6$. Therefore, the equation of the line of intersection is $x = -12t$, $y = 19/12 + 15t$, and $z = 1/6 + 6t$.

Q.4 **Show** that $\displaystyle\lim_{(x,y)\to(0,0)} \left(\frac{x^2 - y^2}{x^2 + y^2}\right)^2$ does not exist.
[SLO 1, PLO 1, Marks 5]

Sol. We approach the point $(0,0)$ along the lines $y = kx$ for different $k \in \mathbb{R}$. We have

$$\lim_{(x,y)\to(0,0)} \left(\frac{x^2 - y^2}{x^2 + y^2}\right)^2 = \lim_{(x,kx)\to(0,0)} \left(\frac{x^2 - k^2x^2}{x^2 + k^2x^2}\right)^2 = \lim_{(x,kx)\to(0,0)} \left(\frac{1 - k^2}{1 + k^2}\right)^2 = \left(\frac{1 - k^2}{1 + k^2}\right)^2.$$

This indicates that along different paths, the values do not agree, therefore, the limit does not exist.

Q.5 **Calculate** all the second partial derivatives of $z := x^2y^2e^{2xy}$. Also verify the equality of the mixed derivatives.
[SLO 3, PLO 1, Marks 5]

Sol. We have

$$\frac{\partial z}{\partial x} = 2xy^2e^{2xy} + 2x^2y^3e^{2xy},$$
$$\frac{\partial^2 z}{\partial x^2} = 2y^2e^{2xy} + 4xy^3e^{2xy} + 4xy^3e^{2xy} + 4x^2y^4e^{2xy},$$
$$\frac{\partial^2 z}{\partial y \partial x} = 4xye^{2xy} + 4x^2y^2e^{2xy} + 6x^2y^2e^{2xy} + 4x^3y^3e^{2xy},$$

$$\frac{\partial z}{\partial y} = 2x^2ye^{2xy} + 2x^3y^2e^{2xy},$$

$$\frac{\partial^2 z}{\partial y^2} = 2x^2e^{2xy} + 4x^3ye^{2xy} + 4x^3ye^{2xy} + 4x^4y^2e^{2xy},$$

$$\frac{\partial^2 z}{\partial x \partial y} = 4xye^{2xy} + 4x^2y^2e^{2xy} + 6x^2y^2e^{2xy} + 4x^3y^3e^{2xy}.$$

Note that

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}.$$

Q.6 **Calculate** the directional derivative of $f(x, y) := e^x \cos(\pi y)$ in the direction of

$$\vec{v} = -\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{j}$$

at point $(0, -1)$.
[SLO 1, PLO 1, Marks 3]

Sol. Since,

$$\nabla f(x, y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} = e^x \cos(\pi y)\mathbf{i} - \pi e^x \sin(\pi y)\mathbf{j}.$$

Therefore,

$$\nabla f(0, -1) = e^0 \cos(-\pi)\mathbf{i} - \pi e^0 \sin(-\pi)\mathbf{j} = -\mathbf{i}.$$

Thus, the directional derivative of f in the direction of $\vec{v} = -\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{j}$ is

$$\vec{v} \cdot \nabla f(0, -1) = (-\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{j}) \cdot (-\mathbf{i}) = \frac{1}{\sqrt{5}}.$$

"Many of life's failures are people who did not realize how close they were to success when they gave up." — Thomas A. Edison