School of Natural Sciences
Department of Mathematics
OHT-1 (MATH-322 Group Theory), Spring 2020
Solution Key

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1. (a) Let $*$ be a binary operation over $\mathbb{Q}^+$ defined by

$$x * y = \frac{xy}{4}, \quad \forall x, y \in \mathbb{Q}^+.$$

Determine whether $(\mathbb{Q}^+, *)$ is a group or not? **(5 Points)**

Ans. Indeed, $(\mathbb{Q}^+, *)$ is a group. Here we show that it satisfies group axioms.
Associativity: For any $x, y, z \in \mathbb{Q}^+$,

$$(x * y) * z = \frac{xy}{4} * z = \frac{(xy/4)z}{4} = \frac{xyz}{16}$$

and likewise

$$x * (y * z) = x * \frac{yz}{4} = \frac{x(yz/4)}{4} = \frac{xyz}{16}.$$

Thus, $*$ is associative on $\mathbb{Q}^+$.
Identity: It is evident that $4 * x = x * 4 = x$ for all $x \in \mathbb{Q}^+$, so $e = 4$ is an identity element for $(\mathbb{Q}^+, *)$.
Inverses: For every element $x \in \mathbb{Q}^+$,

$$x * \frac{16}{x} = \frac{x(16/x)}{4} = 4 = e \quad \text{and} \quad \frac{16}{x} * x = \frac{(16/x)x}{4} = 4 = e.$$

Therefore, $x^{-1} = 4/x$ is the inverse for $x \in \mathbb{Q}^+$ with the operation $*$.

(b) Let $(G, *)$ be a group. Prove that in group G there exist a unique idempotent element. **(5 Points)**

Ans. Let $(G, *)$ be a group. An element $x \in G$ is called idempotent if $x * x = x$. Let e be the identity of the group $(G, *)$. Certainly, e is idempotent, because $e * e = e$. Thus, it only requires to prove uniqueness. Suppose $x \in G$ is idempotent, i.e., $x * x = x$. We show that $x = e$. Towards this end, let x' be the inverse of x in G. Then,

$$\begin{aligned}
(x * x) * x' &= x * x' && \text{(multiply both sides by } x' \text{ on right)}\\
x * (x * x') &= e && \text{(associative and inverse properties)}\\
x * e &= e && \text{(inverse properties)}\\
x &= e && \text{(identity properties)}
\end{aligned}$$

Thus, $x = e$, so G has exactly one idempotent element and it is e.

2. (a) Let $(G, *)$ be a group and $H, K \subset G$ be subgroups of G. Prove that $H \cap K \subset G$ is a subgroup. **(5 Points)**

Ans. Let $(G, *)$ be a group and $H, K \subset G$ be two subgroups of G. To show that $H \cap K$ is a subgroup, we show that it satisfies the closure property, possesses the identity element and contains inverses of all its elements.

Closure Property: Let $a, b \in H \cap K$. It means $a, b \in H$ and $a, b \in K$. Because H and K are subgroups, they are closed so we have $a * b \in H$ and $a * b \in K$. Therefore, $a * b \in H \cap K$.

Identity: Let $e \in G$ be the identity element with respect to the operation $*$. Because H and K are subgroups, they contain e, i.e., $e \in H$ and $e \in K$. So, $e \in H \cap K$.

Inverses: Let $a \in H \cap K$. Since $(G, *)$ is a group, there exists an element $a^{-1} \in G$. Since, H and K are subgroups of G and $a \in H$ and $a \in K$, therefore, $a^{-1} \in H$ and $a^{-1} \in K$. So, $a^{-1} \in H \cap K$.

Hence, $(H \cap K, *)$ is a subgroup of $(G, *)$.

(b) Let $(G, *)$ be a group. Prove that

$$(a * b)^{-1} = b^{-1} * a^{-1}, \quad \forall a, b \in G.$$

(5 Points)

Ans. Let $a, b \in G$ be arbitrary and $a^{-1}, b^{-1} \in G$ denote the inverses of a, b, respectively. By the closure property, $a * b \in G$. Then, by group axioms, there exists an element $(a * b)^{-1} \in G$ that is the inverse of $a * b \in G$, i.e.,

$$(a * b) * (a * b)^{-1} = e.$$

Then, we have

$$\begin{aligned}
a^{-1} * \big((a * b) * (a * b)^{-1}\big) &= a^{-1} * e && \text{(multiply both sides by } a^{-1} \text{ on left)}\\
\big(a^{-1} * (a * b)\big) * (a * b)^{-1} &= a^{-1} && \text{(associative and identity properties)}\\
\big((a^{-1} * a) * b\big) * (a * b)^{-1} &= a^{-1} && \text{(associative property)}\\
(e * b) * (a * b)^{-1} &= a^{-1} && \text{(inverse property)}\\
b * (a * b)^{-1} &= a^{-1} && \text{(identity property)}\\
b^{-1} * \big(b * (a * b)^{-1}\big) &= b^{-1} * a^{-1} && \text{(multiply both sides by } b^{-1} \text{ on left)}\\
\big(b^{-1} * b\big) * (a * b)^{-1} &= b^{-1} * a^{-1} && \text{(associative property)}\\
e * (a * b)^{-1} &= b^{-1} * a^{-1} && \text{(inverse property)}\\
(a * b)^{-1} &= b^{-1} * a^{-1} && \text{(identity property)}
\end{aligned}$$

Hence, proved.

3. (a) Let $(G, *)$, (H, o) and $(M, \bullet)$ be three groups. Let $f : G \to H$ and $g : H \to M$ be homomorphism. Prove that the composition $gf : G \to M$ is homomorphism. **(5 Points)**

Ans. Let $a, b \in G$. For the composite function gf we have

$$\begin{aligned} gf(a*b) &= g(f(a*b)) && \text{(definition of composition)} \\ &= g(f(a) o f(b)) && \text{(homomorphism of } f: G \to H) \\ &= g(f(a)) \bullet g(f(b)) && \text{(homomorphism of } g: H \to M) \\ &= gf(a) \bullet gf(b) && \text{(definition of composition).} \end{aligned}$$

Therefore, $gf : G \to M$ is a homomorphism.

(b) Let $(G, *)$ be a group and $H \subset G$ be subgroup of G. Let us define a relation on G using H as follows.

$$x, y \in G, \quad x \sim y \iff x^{-1} * y \in H. \qquad (*)$$

Prove that (*) gives an equivalence relation on G. **(5 Points)**

Ans. We show that the relation $\sim$ defined in the question is reflexive, symmetric and transitive.

Reflexive: We need to prove that $x \sim x$ for all $x \in G$. Note that $x^{-1} * x = e$ and $e \in H$ as H is a subgroup. Therefore, $x^{-1} * x = e \in H$ for all $x \in G$. Therefore, $x \sim x$ for all $x \in G$.

Symmetric: Let $x, y \in G$ such that $x \sim y$. We show that $y \sim x$. Note that $x \sim y$ implies $x^{-1} * y \in H$. As H is a subgroup, $\left(x^{-1} * y\right)^{-1} \in H$. by Question 2(b), $\left(x^{-1} * y\right)^{-1} = y^{-1} * (x^{-1})^{-1} = y^{-1} * x \in H$. Therefore, $y \sim x$ for all $x, y \in G$.

Transitive: Let $x, y, z \in G$ such that $x \sim y$ and $y \sim z$. Then $x^{-1} * y \in H$ and $y^{-1} * z \in H$. Since H is a subgroup of G, by closure property $(x^{-1} * y) * (y^{-1} * z) \in H$. But,

$$\begin{aligned} (x^{-1} * y) * (y^{-1} * z) &= x^{-1} * \left(y * (y^{-1} * z)\right) && \text{(associative property)} \\ &= x^{-1} * \left((y * y^{-1}) * z\right) && \text{(associative property)} \\ &= x^{-1} * (e * z) && \text{(inverse property)} \\ &= x^{-1} * z && \text{(inverse property).} \end{aligned}$$

Therefore, $x^{-1} * z \in H$ and thus $x \sim z$.

Hence, $\sim$ is an equivalence relation.

"Don't let what you *cannot do* interfere with what you *can do*" — John Wooden.