School of Natural Sciences
Department of Mathematics
OHT-1 (MATH-455 Integral Equations), Spring 2020
Solution Key

1. (a) Classify the integral equations as linear, non-linear, homogeneous, non-homogeneous, singular, non-singular, first kind, second kind, Volterra and Fredholm.
(2.5+2.5 Points)

(i) $1+\dfrac{\varphi(x)}{\cos x}-\lambda\displaystyle\int_0^{\pi/3}\frac{\sin^2(x-t)\varphi(t)}{t^2}dt=0.$

Ans. Linear, non-homogeneous, singular, second kind Fredholm integral equation.

(ii) $2\psi(x)+3\displaystyle\int_0^7 M(x,t)\psi(t)dt=0$, where $M(x,t):=\begin{cases} x^2-t^2, & 0\le t\le x,\\ t^2+x^2, & x\le t\le 7.\end{cases}$

Ans. Linear, homogeneous, non-singular, second kind Fredholm integral equation.

(b) Show that $y(x)=\dfrac{1}{(1+x^2)^{3/2}}$ is a solution to the integral equation

$$y(x)=\frac{1}{(1+x^2)}-\int_0^x\frac{t}{(1+x^2)}y(t)dt. \qquad \textbf{(5 Points)} \tag{1}$$

Ans. Since

$$\begin{aligned}
\text{R.H.S} =& \frac{1}{(1+x^2)}-\int_0^x\frac{ty(t)}{(1+x^2)}dt\\
=& \frac{1}{(1+x^2)}\left[1-\int_0^x\frac{t}{(1+t^2)^{3/2}}dt\right]\\
=& \frac{1}{(1+x^2)}\left[1-\frac{1}{2}\int_0^x 2t(1+t^2)^{-3/2}dt\right]\\
=& \frac{1}{(1+x^2)}\left[1+\left[(1+t^2)^{-1/2}\right]_0^x\right]\\
=& \frac{1}{(1+x^2)}\left[1+\left[(1+x^2)^{-1/2}-1\right]\right]\\
=& \frac{1}{(1+x^2)(1+x^2)^{1/2}}\\
=& y(x)\\
=& \text{L.H.S.},
\end{aligned}$$

therefore, $y(x)=\dfrac{1}{(1+x^2)^{3/2}}$ is a solution to the integral equation (1).

2. Find the spectrum of the integral equation

$$g(x)=\lambda\int_{-\pi}^{\pi}x\sin t\, g(t)dt, \tag{2}$$

and eigen solutions. Discuss qualities of the spectrum (any two).

Ans. Note that equations (2) is a Fredholm equation with separable kernel $K(x,t) = x \sin t$. Therefore, the solution is

$$g(x) = C\lambda x, \tag{a}$$

where

$$C := \int_{-\pi}^{\pi} \sin t g(t) dt.$$

To find the value of C, multiply equation (2) with $\sin x$ and integrate over $(-\pi, \pi)$. This yields,

$$C = \lambda \int_{-\pi}^{\pi} x \sin x dx \int_{-\pi}^{\pi} \sin t g(t) dt = C\lambda \int_{-\pi}^{\pi} x \sin x dx,$$

or simply

$$C \left[1 - \lambda \int_{-\pi}^{\pi} x \sin x dx \right] = 0.$$

Since,

$$\int_{-\pi}^{\pi} x \sin x dx = \Big[-x \cos x + \sin x \Big]_{-\pi}^{\pi} = -\pi \cos \pi + \sin \pi - \pi \cos(-\pi) - \sin(-\pi) = 2\pi,$$

we have

$$C \left[1 - 2\pi\lambda \right] = 0.$$

Therefore, for non-trivial solutions, we must have $\lambda = 1/2\pi$. As this is the only possible value of λ rendering a non-trivial solution, the spectrum of (2) consists of one element $\lambda = 1/2\pi$. Moreover, since the only solution associated to eigenvalue $\lambda = 1/2\pi$ is $g(x) = Cx/2\pi$, the multiplicity of the eigenvalue is 1. Thus, the spectrum of the integral equation (2) is finite and discrete, and the only element has multiplicity 1.

(6+2+2 Points)

3. Solve and identify the resolvent kernel of the integral equation

$$g(x) = x + \lambda \int_{-\pi}^{\pi} x \sin t \, g(t) dt. \qquad \textbf{(7+3 Points)} \tag{3}$$

Ans. Note that equation (3) is a Fredholm with separable kernel $K(x,t) = x \sin t$. Therefore, the solution is

$$g(x) = x + C\lambda x, \tag{b}$$

with

$$C := \int_{-\pi}^{\pi} \sin t g(t) dt.$$

To find the value of C, multiply equation (3) with $\sin x$ and integrate over $(-\pi, \pi)$. This yields,

$$C = \int_{-\pi}^{\pi} t \sin t dt + \lambda \int_{-\pi}^{\pi} x \sin x dx \int_{-\pi}^{\pi} \sin t g(t) dt.$$

or simply

$$C \left[1 - \lambda \int_{-\pi}^{\pi} x \sin x dx \right] = \int_{-\pi}^{\pi} t \sin t dt.$$

As calculated in the previous question, $\int_{\pi}^{\pi} x \sin x dx = 2\pi$. Therefore, for $\lambda \neq 1/2\pi$, we have

$$C = \frac{1}{1 - 2\pi\lambda} \int_{-\pi}^{\pi} x \sin t g(t) dt.$$

Substituting the value of C in Eq. (b), we arrive at the solution in the form

$$g(x) = x + \int_{-\pi}^{\pi} \left[\frac{\lambda x \sin t}{1 - 2\pi\lambda} \right] t \, dt = x + \int_{-\pi}^{\pi} R(x, t; \lambda) \, t \, dt,$$

where $R(x, t; \lambda) := \dfrac{\lambda x \sin t}{1 - 2\pi\lambda}$ is the resolvent kernel. On simplification, we arrive at the solution

$$g(x) = x + \frac{2\pi\lambda}{1 - 2\pi\lambda} x = \frac{x}{1 - 2\pi\lambda}.$$

4. Convert the integral equation

$$u(\xi) = \lambda \int_{0}^{1} \kappa(\xi, t) u(t) dt, \tag{4}$$

with

$$\kappa(\xi, t) = \begin{cases} \xi(1 - t), & \xi \leq t \leq 1, \\ t(1 - \xi), & 0 \leq t \leq \xi, \end{cases} \tag{5}$$

to a boundary value problem with suitable boundary conditions. **(8+2 Points)**

Ans. Remark that the integral equation (4) together with (5) can be rewritten as

$$u(\xi) = \lambda \int_{0}^{\xi} t(1 - \xi) u(t) dt + \lambda \int_{\xi}^{1} \xi(1 - t) u(t) dt. \tag{c}$$

Differentiating both sides with respect to ξ, we get

$$\begin{aligned} u'(\xi) =& \lambda\xi(1-\xi)u(\xi) + \lambda\int_0^\xi \frac{\partial}{\partial\xi}(1-\xi)tu(t)dt - \lambda\xi(1-\xi)u(\xi)u(\xi) + \lambda\int_1^\xi \frac{\partial}{\partial\xi}\xi(1-t)u(t)dt \\ =& -\lambda\int_0^\xi tu(t)dt + \lambda\int_\xi^1 (1-t)u(t)dt. \end{aligned}$$

Differentiating again, we get

$$u''(\xi) = -\lambda\left[\xi u(\xi) - 0 + 0\right] + \lambda\left[0 - (1-\xi)u(\xi) + 0\right] = -\lambda u(\xi).$$

Therefore, $u(\xi)$ satisfies the second order ordinary differential equation

$$u''(\xi) + \lambda u(\xi) = 0.$$

For boundary conditions, we consider equation (c) and substitute $\xi = 0$ and $\xi = 1$ to get, respectively,

$$u(0) = 0 \qquad \text{and} \qquad u(1) = 0.$$

Therefore, integral equation (4) can be converted to the boundary value problem

$$\begin{cases} u''(\xi) + \lambda u(\xi) = 0, & \xi \in (0,1), \\ u(0) = 0 = u(1). \end{cases}$$

5. Using the potential function $\varphi(x) := \frac{d^2v}{dx^2}$, form an integral equation corresponding to the initial value problem

$$\frac{d^2v}{dx^2} + x\frac{dv}{dx} + v = 0, \tag{6}$$
$$v(0) = 1, \tag{7}$$
$$\frac{dv}{dx}(0) = 0. \tag{8}$$

(10 Points)

Ans. Let

$$\varphi(x) := \frac{d^2v}{dx^2}. \tag{d}$$

Then, integrating (d) over the interval $(0, x)$, and using the first fundamental theorem of Calculus, one arrives at

$$\frac{dv}{dx}(x) - \frac{dv}{dx}(0) = \int_0^x \varphi(t)dt.$$

Invoking (8), one finds out that

$$\frac{dv}{dx} = \int_0^x \varphi(t)dt. \tag{e}$$

Once again, integrating (e) over the interval $(0, x)$, and using the first fundamental theorem of Calculus, one arrives at

$$v(x) - v(0) = \int_0^x \int_0^t \varphi(\xi)d\xi\, dt = \int_0^x (x-t)\varphi(t)dt.$$

This time we invoke (8) to eliminate $v(0)$, which yields

$$v(x) = 1 + \int_0^x (x-t)\varphi(t)dt. \tag{f}$$

Together with Eqs. (d), (e), and (f), differential equation (6) furnishes

$$\varphi(x) + x\left[\int_0^x \varphi(t)dt\right] + \left[1 + \int_0^x (x-t)\varphi(t)dt\right] = 0.$$

or equivalently,

$$\varphi(x) = -1 - \int_0^x (2x-t)\varphi(t)dt.$$

"Don't let what you *cannot do* interfere with what you *can do*" — John Wooden.