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On the Green function in visco-elastic media obeying a frequency power-law

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In this work, we present an explicit expression for the Green function in a visco-elastic medium. We choose Szabo and Wu's frequency power law model to describe the visco-elastic properties and derive a generalized visco-elastic wave equation. We express the ideal Green function (without any viscous effect) in terms of the viscous Green function using an attenuation operator. By means of an approximation of the ideal Green function, we address the problem of reconstructing a small anomaly in a visco-elastic medium from wavefield measurements. Copyright © 2011 John Wiley & Sons, Ltd.

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1. Introduction

The elastic properties of human soft tissues have been exploited in a number of imaging modalities in the recent past, because the elasticity properties vary significantly in order of magnitude with different tissue types and are closely linked with the pathology of the tissues and their underlying structure.

Most of the time, medium is considered to be ideal (without any viscous effect), neglecting the fact that a wave loses some of its energy to the medium and its amplitude decreases with time due to viscosity. An estimation of the viscosity effects, however, can sometimes be very useful in the characterization and identification of anomaly [1].

To address the problem of reconstructing a small anomaly in visco-elastic media from wavefield measurements, it is important to first model the mechanical response of such media to excitations.

The Voigt model is a common model to describe the visco-elastic properties of tissues. Catheline *et al.* [2] have shown that this model is well adapted to describe the visco-elastic response of tissues to low-frequency excitations. However, we choose a more general model derived by Szabo and Wu in [3] that describes observed power-law behavior of many visco-elastic materials including human myocardium. This model is based on a time-domain statement of causality [4] and reduces to the Voigt model for the specific case of quadratic frequency losses.

Expressing the ideal elastic field without any viscous effect in terms of the measured field in a viscous medium, one can generalize the methods described in [5–9], namely the time reversal, back-propagation and Kirchhoff imaging, to recover the visco-elastic and geometric properties of an anomaly from wavefield measurements. To achieve this goal, we focus on the Green function in this article. We identify a relationship between the ideal Green function and the visco-elastic Green function in the limiting case when the shear modulus $\lambda \to \infty$, in a quasi-incompressible medium. We also provide an approximation of this relationship using the stationary phase theorem.

The article is organized as follows. In Section 2, we introduce a general visco-elastic wave equation based on Szabo and Wu's power law model. Section 3 is devoted to the derivation of the Green function in the visco-elastic medium. In Section 4, we approximate the ideal Green function in the case of quadratic losses and sketch a procedure of image reconstruction in visco-elastic media. We support our work with numerical illustrations, which are presented in Section 5.

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2. General visco-elastic wave equation

When a wave travels through a biological medium, its amplitude decreases with time due to attenuation. The attenuation coefficient for biological tissues may be approximated by a power-law over a wide range of frequencies. Measured attenuation coefficients of soft tissues typically have linear or greater than linear dependence on frequency [3, 4, 10].

In a pure elastic medium; without attenuation, Hooke's law states that:

$$\mathbb{T}(x,t)=\mathbb{C}:\mathbb{S}(x,t),$$

where $x\in\mathbb{R}^3$, t is the time variable, $\mathbb{T}$ is the order two stress tensor, $\mathbb{C}$ is the order four stiffness tensor and $\mathbb{S}=\frac{1}{2}(\nabla\mathbf{u}+\nabla\mathbf{u}^{\mathrm{T}})$ is the order 2 strain tensor. Here T represents the transpose operation, ' : ' represents tensorial product and $\mathbf{u}(x,t)$ is the displacement field.

Consider a visco-elastic medium. Suppose that the medium is homogeneous and isotropic. We write

$$\mathbb{C}=[\mathscr{C}_{\mathrm{ijkl}}]=[\lambda\delta_{ij}\delta_{kl}+\mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})],$$
$$\mathbb{C}^V=[\eta_{\mathrm{ijkl}}]=[\eta_s\delta_{ij}\delta_{kl}+\eta_p(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})],$$

where $\mathbb{C}^V$ is the order four viscosity tensor, δ_{ab} is the Kronecker delta function, (μ,λ) are the Lamé parameters, and (η_s,η_p) are the shear and bulk viscosities, respectively. Throughout this work, we suppose that

$$\eta_p,\eta_s\ll 1. \tag{1}$$

For a medium obeying a power-law attenuation model, under the smallness condition (1), a generalized Hooke's law reads [3]

$$\mathbb{T}(x,t)=\mathbb{C}:\mathbb{S}(x,t)+\mathbb{C}^V:\mathscr{M}[\mathbb{S}](x,t), \tag{2}$$

where $\mathscr{M}$ is the convolution operator given by:

$$\mathscr{M}[\mathbb{S}]=\begin{cases}-(-1)^{\gamma/2}\dfrac{\partial^{\gamma-1}\mathbb{S}}{\partial t^{\gamma-1}}, & \gamma \text{ is an even integer,}\\ \dfrac{2}{\pi}(\gamma-1)!(-1)^{(\gamma+1)/2}\dfrac{H(t)}{t^{\gamma}}*_t\mathbb{S}, & \gamma \text{ is an odd integer,}\\ -\dfrac{2}{\pi}\Gamma(\gamma)\sin(\gamma\pi/2)\dfrac{H(t)}{|t|^{\gamma}}*_t\mathbb{S}, & \gamma \text{ is a non integer.}\end{cases} \tag{3}$$

Here $H(t)$ is the Heaviside function, Γ is the gamma function and $*_t$ represents the convolution with respect to t.

Remark 2.1
Note that for the common case when, $\gamma=2$, the generalized Hooke's law (2) reduces to the Voigt model,

$$\mathbb{T}=\mathbb{C}:\mathbb{S}+\mathbb{C}^V:\frac{\partial\mathbb{S}}{\partial t}.$$

To find a general visco-elastic wave equation, we take the divergence of (2) which gives

$$\nabla\cdot\mathbb{T}=(\bar{\lambda}+\bar{\mu})\nabla(\nabla\cdot\mathbf{u})+\bar{\mu}\Delta\mathbf{u}, \tag{4}$$

and substitute the resulting expression (4) in the equation of motion (5) for the system, i.e.

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2}-\mathbf{F}=\nabla\cdot\mathbb{T}. \tag{5}$$

We obtain the generalized visco-elastic wave equation

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2}-\mathbf{F}=(\bar{\lambda}+\bar{\mu})\nabla(\nabla\cdot\mathbf{u})+\bar{\mu}\Delta\mathbf{u}, \tag{6}$$

where ρ is the density (assumed to be constant here), $\mathbf{F}$ is the applied force and

$$\bar{\lambda}=\lambda+\eta_p\mathscr{M}[\cdot]\quad\text{and}\quad\bar{\mu}=\mu+\eta_s\mathscr{M}[\cdot].$$

3. Green function

In this section we find the Green function for the visco-elastic wave equation (6). We first derive the following Helmholtz decomposition of the displacement field:

Lemma 3.1
If the displacement field $\mathbf{u}(x,t)$ satisfy (6) such that $(\partial \mathbf{u}/\partial t)(x,0)=\nabla A+\nabla\times\vec{B}$ and $\mathbf{u}(x,0)=\nabla C+\nabla\times\vec{D}$ where $\nabla\cdot\vec{B}=0=\nabla\cdot\vec{D}$ and if the body force $\mathbf{F}=\nabla\varphi_f+\nabla\times\vec{\psi}_f$ with $\nabla\cdot\vec{\psi}_f=0$ then there exist potentials φ_u and $\vec{\psi}_u$ such that

- $\mathbf{u}=\nabla\varphi_u+\nabla\times\vec{\psi}_u$; $\nabla\cdot\vec{\psi}_u=0$;
- $(\partial^2/\partial t^2)\varphi_u=(1/\rho)\varphi_f+c_p^2\Delta\varphi_u+\nu_p\mathscr{M}[\Delta\varphi_u]\approx(1/\rho)\varphi_f-(\nu_p/\rho c_p^2)\mathscr{M}[\varphi_f]+c_p^2\Delta\varphi_u+(\nu_p/c_p^2)\mathscr{M}[\partial_t^2\varphi_u]$;
- $(\partial^2/\partial t^2)\vec{\psi}_u=(1/\rho)\vec{\psi}_f+c_s^2\Delta\vec{\psi}_u+\nu_s\mathscr{M}[\Delta\vec{\psi}_u]\approx(1/\rho)\vec{\psi}_f-(\nu_s/\rho c_s^2)\mathscr{M}[\vec{\psi}_f]+c_s^2\Delta\vec{\psi}_u+(\nu_s/c_s^2)\mathscr{M}[\partial_t^2\vec{\psi}_u]$,

with

$$c_p^2=\frac{\lambda+2\mu}{\rho},\quad c_s^2=\frac{\mu}{\rho},\quad \nu_p=\frac{\eta_p+2\eta_s}{\rho}\quad\text{and}\quad \nu_s=\frac{\eta_s}{\rho}.$$

Proof
For φ_u and $\vec{\psi}_u$ defined as:

$$\varphi_u(x,t)=\int_0^t\int_0^\tau\left(\frac{1}{\rho}\varphi_f+(c_p^2+\nu_p\mathscr{M})[\nabla\cdot u]\right)\mathrm{d}s\,\mathrm{d}\tau+tA+C, \tag{7}$$

$$\vec{\psi}_u(x,t)=\int_0^t\int_0^\tau\left(\frac{1}{\rho}\vec{\psi}_f-(c_s^2+\nu_s\mathscr{M})[\nabla\times u]\right)\mathrm{d}s\,\mathrm{d}\tau+t\vec{B}+\vec{D}. \tag{8}$$

We have the required expression for $\mathbf{u}$. Moreover, it is evident from (8) that $\nabla\cdot\psi_u=0$

Now, on differentiating φ_u and $\vec{\psi}_u$ twice with respect to time, we obtain

$$\frac{\partial^2\varphi_u}{\partial t^2}=\frac{1}{\rho}\varphi_f+c_p^2\Delta\varphi_u+\nu_p\mathscr{M}[\Delta\varphi_u],$$

$$\frac{\partial^2\vec{\psi}_u}{\partial t^2}=\frac{1}{\rho}\vec{\psi}_f+c_s^2\Delta\vec{\psi}_u+\nu_s\mathscr{M}[\Delta\vec{\psi}_u].$$

Finally we invoke (1). By applying $\mathscr{M}$ on last two equations, neglecting the higher order terms in ν_s and ν_p and injecting back the expressions for $\mathscr{M}[\Delta\varphi_u]$ and $\mathscr{M}[\Delta\psi_u]$, we get the required differential equations for φ_u and ψ_u. □

Let

$$K_m(\omega)=\omega\sqrt{\left(1-\frac{\nu_m}{c_m^2}\hat{\mathscr{M}}[\omega]\right)},\quad m=s,p, \tag{9}$$

where the multiplication operator $\hat{\mathscr{M}}[\omega]$ is the Fourier transform of the convolution operator $\mathscr{M}$ and ω is the frequency. If φ_u and ψ_u are causal, then it implies the causality of the inverse Fourier transform of $K_m(\omega), m=s,p$. Applying the Kramers–Krönig relations[‡], it follows that

$$-\Im m K_m(\omega)=\mathscr{H}[\Re e K_m(\omega)]\quad\text{and}\quad \Re e K_m(\omega)=\mathscr{H}[\Im m K_m(\omega)],\quad m=p,s, \tag{10}$$

where $\mathscr{H}$ is the Hilbert transform, $\Im$ and $\Re$ represent the imaginary and the real parts of a complex number, respectively. Recall that $\mathscr{H}^2=-I$. The convolution operator $\mathscr{M}$ given by (3) is based on the constraint that causality imposes on (2). Under the smallness assumption (1), the expressions in (3) can be found from the Kramers–Krönig relations (10). One drawback of (10) is that the attenuation, $\Im m K_m(\omega)$, must be known at all frequencies to determine the dispersion, $\Re e K_m(\omega)$. However, bounds on the dispersion can be obtained from measurements of the attenuation over a finite frequency range [12].

3.1. Solution of (6) with a concentrated force

Let u_{ij} denote the ith component of the solution $\mathbf{u}_j$ of the elastic wave equation (6) related to a force $\mathbf{F}$ concentrated in the x_j-direction. Let $j=1$ for simplicity and suppose that

$$\mathbf{F}=-T(t)\delta(x-\xi)\mathbf{e}_1=-T(t)\delta(x-\xi)(1,0,0), \tag{11}$$

where ξ is the source point and $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$ is an orthonormal basis of $\mathbb{R}^3$.

‡ *See [4, 3, 11] for more details on causality and KKR.*

Let $\mathbf{Z}$ be the solution of the poisson equation

$$\nabla^2 \mathbf{Z} = \mathbf{F}.$$

Then

$$\mathbf{Z}(x,t;\xi) = \frac{T(t)}{4\pi}\frac{1}{r}\mathbf{e}_1.$$

As $\nabla^2\mathbf{Z} = \nabla(\nabla\cdot\mathbf{Z}) - \nabla\times(\nabla\times\mathbf{Z})$, the Helmholtz decomposition of the force $\mathbf{F}$ can be written [13] as

$$\begin{cases} \mathbf{F} = \nabla\varphi_f + \nabla\times\psi_f, \\ \varphi_f = \nabla\cdot\mathbf{Z} = \dfrac{T(t)}{4\pi}\dfrac{\partial}{\partial x_1}\left(\dfrac{1}{r}\right), \\ \psi_f = -\nabla\times\mathbf{Z} = -\dfrac{T(t)}{4\pi}\left(0, \dfrac{\partial}{\partial x_3}\left(\dfrac{1}{r}\right), -\dfrac{\partial}{\partial x_2}\left(\dfrac{1}{r}\right)\right), \end{cases} \tag{12}$$

where $r = |x-\xi|$.

Consider the Helmholtz decomposition for $\mathbf{u}_1$ as

$$\mathbf{u}_1 = \nabla\varphi_1 + \nabla\times\psi_1 \tag{13}$$

then, according to Lemma 3.1, φ_1 and ψ_1 are, respectively, the solutions of the equations

$$\Delta\varphi_1 - \frac{1}{c_p^2}\frac{\partial^2\varphi_1}{\partial t^2} + \frac{v_p}{c_p^4}\mathscr{M}[\partial_t^2\varphi_1] = \frac{v_p}{\rho c_p^4}\mathscr{M}[\varphi_f] - \frac{1}{c_p^2\rho}\varphi_f, \tag{14}$$

$$\Delta\psi_1 - \frac{1}{c_s^2}\frac{\partial^2\psi_1}{\partial t^2} + \frac{v_s}{c_s^4}\mathscr{M}[\partial_t^2\psi_1] = \frac{v_s}{\rho c_s^4}\mathscr{M}[\psi_f] - \frac{1}{c_s^2\rho}\psi_f. \tag{15}$$

Taking the Fourier transform of (13), (14) and (15) with respect to t we get

$$\hat{\mathbf{u}}_1 = \nabla\hat{\varphi}_1 + \nabla\times\hat{\psi}_1, \tag{16}$$

$$\Delta\hat{\varphi}_1 + \frac{1}{c_p^2}K_p^2(\omega)\hat{\varphi}_1 = \frac{v_p}{\rho c_p^4}\hat{\mathscr{M}}[\omega]\hat{\varphi}_f - \frac{1}{\rho c_p^2}\hat{\varphi}_f, \tag{17}$$

$$\Delta\hat{\psi}_1 + \frac{1}{c_s^2}K_s^2(\omega)\hat{\psi}_1 = \frac{v_s}{\rho c_s^4}\hat{\mathscr{M}}[\omega]\hat{\psi}_f - \frac{1}{\rho c_s^2}\hat{\psi}_f, \tag{18}$$

where $K_m(\omega)$, $m = p,s$, are defined in (9).

We remind that the Green function of the Helmholtz equations (17) and (18) is

$$\hat{g}^m(x,\omega) = \frac{e^{\sqrt{-1}\frac{K_m(\omega)}{c_m}|x|}}{4\pi|x|}, \quad m = s,p.$$

We closely follow the arguments in [13], and write $\hat{\varphi}_1$ as:

$$\begin{aligned}\hat{\varphi}_1(x,\omega;\xi) &= \hat{g}^m(x,\omega) *_x \left(\frac{v_p}{\rho c_p^4}\hat{\mathscr{M}}[\omega]\varphi_f - \frac{1}{c_p^2\rho}\varphi_f\right) \\ &= -\left(1 - \frac{v_p}{c_p^2}\hat{\mathscr{M}}[\omega]\right)\frac{\hat{T}(\omega)}{\rho(4\pi c_p)^2}\int_{\mathbb{R}^3}\hat{g}^p(x-z,\omega)\frac{\partial}{\partial z_1}\frac{1}{|z-\xi|}\,\mathrm{d}z.\end{aligned}$$

Remark that $z \to \hat{g}^p(x-z,\omega)$ is constant on each sphere $\partial B(x,h)$, centered at x with radius h. Therefore, use of spherical coordinates leads to

$$\hat{\varphi}_1(x,\omega;\xi) = -\left(1 - \frac{v_p}{c_p^2}\hat{\mathscr{M}}[\omega]\right)\frac{1}{\rho(4\pi c_p)^2}\hat{T}(\omega)\int_0^\infty \hat{g}^p(h,\omega)\int_{\partial B(x,h)}\frac{\partial}{\partial z_1}\left(\frac{1}{|z-\xi|}\right)\mathrm{d}\sigma(z)\,\mathrm{d}h$$

where $\mathrm{d}\sigma(z)$ is the surface element on $\partial B(x,h)$.

From [14], it follows that

$$\int_{\partial B(x,h)}\frac{\partial}{\partial z_1}\left(\frac{1}{|z-\xi|}\right)\mathrm{d}\sigma(z) = \begin{cases} 0 & \text{if } h > r \\ 4\pi h^2\dfrac{\partial}{\partial x_1}\left(\dfrac{1}{r}\right) & \text{if } h < r. \end{cases}$$

Therefore, we have the following expression for $\hat{\varphi}_1$:

$$\hat{\varphi}_1(x,\omega;\xi) = -\left(1-\frac{v_p}{c_p^2}\hat{\mathscr{M}}[\omega]\right)\frac{1}{4\pi\rho c_p^2}\hat{T}(\omega)\frac{\partial}{\partial x_1}\left(\frac{1}{r}\right)\int_0^r h e^{\sqrt{-1}\frac{K_p(\omega)}{c_p}h}\,\mathrm{d}h,$$

$$= -\left(1-\frac{v_p}{c_p^2}\hat{\mathscr{M}}[\omega]\right)\frac{1}{4\pi\rho}\hat{T}(\omega)\frac{\partial}{\partial x_1}\left(\frac{1}{r}\right)\int_0^{r/c_p}\zeta e^{\sqrt{-1}K_p(\omega)\zeta}\,\mathrm{d}\zeta. \tag{19}$$

In the same way, the vector $\hat{\psi}_1$ is given by:

$$\hat{\psi}_1(x,\omega;\xi) = \left(1-\frac{v_s}{c_s^2}\hat{\mathscr{M}}[\omega]\right)\frac{1}{4\pi\rho}\hat{T}(\omega)\left(0,\frac{\partial}{\partial x_3}\left(\frac{1}{r}\right),-\frac{\partial}{\partial x_2}\left(\frac{1}{r}\right)\right)\int_0^{r/c_s}\zeta e^{\sqrt{-1}K_s(\omega)\zeta}\,\mathrm{d}\zeta. \tag{20}$$

Here we introduce the following notations for simplicity:

$$I_m(r,\omega) = A_m\int_0^{r/c_m}\zeta e^{\sqrt{-1}K_m(\omega)\zeta}\,\mathrm{d}\zeta \tag{21}$$

$$E_m(r,\omega) = A_m e^{\sqrt{-1}K_m(\omega)\frac{r}{c_m}}, \tag{22}$$

$$A_m(\omega) = \left(1-\frac{v_m}{c_m^2}\hat{\mathscr{M}}[\omega]\right), \quad m=p,s. \tag{23}$$

and calculate $\hat{u}_{i1} = (\nabla\varphi_1)_i + (\nabla\times\vec{\psi}_1)_i$. For all $i=1:3$

$$(\nabla\hat{\varphi}_1)_i = -\frac{\partial}{\partial x_i}\left[\left(1-\frac{v_p}{c_p^2}\hat{\mathscr{M}}[\omega]\right)\frac{1}{4\pi\rho}\hat{T}(\omega)\frac{\partial}{\partial x_1}\left(\frac{1}{r}\right)\int_0^{r/c_p}\zeta e^{\sqrt{-1}K_p(\omega)\zeta}\,\mathrm{d}\zeta\right],$$

$$= -\left(1-\frac{v_p}{c_p^2}\hat{\mathscr{M}}[\omega]\right)\frac{1}{4\pi\rho}\hat{T}(\omega)\frac{\partial^2}{\partial x_1 x_i}\left(\frac{1}{r}\right)\int_0^{r/c_p}\zeta e^{\sqrt{-1}K_p(\omega)\zeta}\,\mathrm{d}\zeta - \left(1-\frac{v_p}{c_p^2}\hat{\mathscr{M}}[\omega]\right)\frac{1}{4\pi\rho}\hat{T}(\omega)\frac{\partial}{\partial x_1}\left(\frac{1}{r}\right)\frac{\partial r}{\partial x_i}\left(\frac{r}{c_p^2}e^{\sqrt{-1}K_p(\omega)\frac{r}{c_p}}\right),$$

$$= -\frac{1}{4\pi\rho}\hat{T}(\omega)\frac{\partial^2}{\partial x_i\partial x_1}\left(\frac{1}{r}\right)I_p(r,\omega) + \frac{1}{4\pi\rho c_p^2 r}\hat{T}(\omega)\frac{\partial r}{\partial x_1}\frac{\partial r}{\partial x_i}E_p(r,\omega),$$

where we have used the equality $(\partial/\partial x_1)(1/r) = -(1/r^2)\partial r/\partial x_1$.

Similarly, the value $(\nabla\times\hat{\psi}_1)_i$ is given by:

$$\left(\nabla\times\hat{\psi}_1\right)_i = \frac{1}{4\pi\rho}\hat{T}(\omega)\frac{\partial^2}{\partial x_i\partial x_1}\left(\frac{1}{r}\right)I_s(r,\omega) + \frac{1}{4\pi\rho c_s^2 r}\hat{T}(\omega)\left(\delta_{i1}-\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_1}\right)E_s(r,\omega).$$

Therefore,

$$\hat{u}_{i1} = \frac{1}{4\pi\rho}\hat{T}(\omega)\frac{\partial^2}{\partial x_i x_1}\left(\frac{1}{r}\right)\left[I_s(r,\omega)-I_p(r,\omega)\right] + \frac{1}{4\pi\rho c_p^2 r}\hat{T}(\omega)\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_1}E_p(r,\omega) + \frac{1}{4\pi\rho c_s^2 r}\hat{T}(\omega)\left(\delta_{i1}-\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_1}\right)E_s(r,\omega).$$

Hence, $\hat{u}_{ij}$, the ith component of the solution $\hat{\mathbf{u}}_j$ for an arbitrary j is

$$\hat{u}_{ij} = \frac{1}{4\pi\rho}\hat{T}(\omega)\left(3\gamma_i\gamma_j-\delta_{ij}\right)\frac{1}{r^3}\left[I_s(r,\omega)-I_p(r,\omega)\right] + \frac{1}{4\pi\rho c_p^2}\hat{T}(\omega)\gamma_i\gamma_j\frac{1}{r}E_p(r,\omega) + \frac{1}{4\pi\rho c_s^2}\hat{T}(\omega)\left(\delta_{ij}-\gamma_i\gamma_j\right)\frac{1}{r}E_s(r,\omega),$$

where $\gamma_i = (\partial r/\partial x_i) = (x_i-\xi_i)/r$ and I_m and E_m are given by Equations (21) and (22).

3.2. *Visco-elastic green function*

If we substitute $T(t)=\delta(t)$, where δ is the Dirac delta function, then $\hat{T}(\omega)=1$. Let G_{ij} be the ith component of the Green function related to the force concentrated in the x_j-direction and $\hat{G}_{ij}$ be the Fourier transform of G_{ij} then we have the following expression for $\hat{G}_{ij}$:

$$\hat{G}_{ij}(x,\omega;\xi) = \frac{1}{4\pi\rho}\left(3\gamma_i\gamma_j-\delta_{ij}\right)\frac{1}{r^3}\left[I_s(r,\omega)-I_p(r,\omega)\right] + \frac{1}{4\pi\rho c_p^2}\gamma_i\gamma_j\frac{1}{r}E_p(r,\omega) + \frac{1}{4\pi\rho c_s^2}\left(\delta_{ij}-\gamma_i\gamma_j\right)\frac{1}{r}E_s(r,\omega),$$

or equivalently,

$$\hat{G}_{ij}(x,\omega;\xi) = \hat{g}_{ij}^p(x,\omega;\xi) + \hat{g}_{ij}^s(x,\omega;\xi) + \hat{g}_{ij}^{ps}(x,\omega;\xi), \tag{24}$$

where

$$\hat{g}^{ps}_{ij}(x,\omega;\xi) = \frac{1}{4\pi\rho}(3\gamma_i\gamma_j - \delta_{ij})\frac{1}{r^3}[I_s(r,\omega) - I_p(r,\omega)], \tag{25}$$

$$\hat{g}^{p}_{ij}(x,\omega;\xi) = \frac{1}{\rho c_p^2}A_p(\omega)\gamma_i\gamma_j\hat{g}^p(r,\omega), \tag{26}$$

and

$$\hat{g}^{s}_{ij}(x,\omega;\xi) = \frac{1}{\rho c_s^2}A_s(\omega)(\delta_{ij} - \gamma_i\gamma_j)\hat{g}^s(r,\omega). \tag{27}$$

Let $\mathbb{G}(x,t;\xi) = (G_{ij}(x,t;\xi))_{i,j=1}^3$ denote the transient Green function of (6) associated with the source point ξ. Let $G^m(r,t)$ and $W_m(x,t)$ be the inverse Fourier transforms of $A_m(\omega)\hat{g}^m(r,\omega)$ and $I_m(r,\omega), m=p,s$, respectively. Then, from (24)–(27), we have

$$G_{ij}(x,t;\xi) = \frac{1}{\rho c_p^2}\gamma_i\gamma_j G^p(r,t) + \frac{1}{\rho c_s^2}(\delta_{ij} - \gamma_i\gamma_j)G^s(r,t) + \frac{1}{4\pi\rho}(3\gamma_i\gamma_j - \delta_{ij})\frac{1}{r^3}[W_s(r,t) - W_p(r,t)]. \tag{28}$$

Remark that by a change of variables,

$$W_m(r,t) = \frac{4\pi}{c_m^2}\int_0^r \zeta^2 G^m(\zeta,t;\xi)\,\mathrm{d}\zeta.$$

4. Approximate green function for voigt model

Consider the limiting case $\lambda \to +\infty$. The Green function for a quasi-incompressible visco-elastic medium is given by:

$$G_{ij}(x,t;\xi) = \frac{1}{\rho c_s^2}(\delta_{ij} - \gamma_i\gamma_j)G^s(r,t) + \frac{1}{\rho c_s^2}(3\gamma_i\gamma_j - \delta_{ij})\frac{1}{r^3}\int_0^r \zeta^2 G^s(\zeta,t)\,\mathrm{d}\zeta.$$

To generalize the detection algorithms presented in [5–8] to the visco-elastic case we shall express the ideal Green function without any viscous effect in terms of the Green function in a viscous medium. From

$$G^s(r,t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{e}^{-\sqrt{-1}\omega t}A_s(\omega)\hat{g}^s(r,\omega)\,\mathrm{d}\omega,$$

it follows that

$$G^s(r,t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} A_s(\omega)\frac{\mathrm{e}^{\sqrt{-1}\left(-\omega t + \frac{K_s(\omega)}{c_s}r\right)}}{4\pi r}\,\mathrm{d}\omega.$$

Let us introduce the operator

$$L[\phi(t)] = \frac{1}{2\pi}\int_{\mathbb{R}}\int_0^{+\infty} A_s(\omega)\phi(\tau)\mathrm{e}^{\sqrt{-1}K_s(\omega)\tau}\mathrm{e}^{-\sqrt{-1}\omega t}\,\mathrm{d}\tau\,\mathrm{d}\omega,$$

for a causal function ϕ. Then we have

$$G^s(r,t;\xi) = L\left[\frac{\delta(\tau - r/c_s)}{4\pi r}\right],$$

and therefore,

$$L^*[G^s](r,t) = L^*L\left[\frac{\delta(\tau - r/c_s)}{4\pi r}\right],$$

where L^* is the $L^2(0,+\infty)$-adjoint of L.

Consider for simplicity the Voigt model. Then, $\hat{\mathscr{M}}(\omega) = -\sqrt{-1}\omega$ and hence,

$$K_s(\omega) = \omega\sqrt{1 + \frac{\sqrt{-1}\nu_s}{c_s^2}\omega} \approx \omega + \frac{\sqrt{-1}\nu_s}{2c_s^2}\omega^2,$$

under the smallness condition (1). The operator L can then be approximated by

$$\tilde{L}[\phi](t) = \frac{1}{2\pi}\int_{\mathbb{R}}\int_0^{+\infty} A_s(\omega)\phi(\tau)\mathrm{e}^{-\frac{\nu_s}{2c_s^2}\omega^2\tau}\mathrm{e}^{\sqrt{-1}\omega(\tau - t)}\,\mathrm{d}\tau\,\mathrm{d}\omega.$$

Since

$$\int_{\mathbb{R}} e^{-\frac{v_S}{2c_S^2}\omega^2\tau} e^{\sqrt{-1}\omega(\tau-t)}\, d\omega = \frac{\sqrt{2\pi}c_S}{\sqrt{v_S\tau}} e^{-\frac{c_S^2(\tau-t)^2}{2v_S\tau}},$$

and

$$\sqrt{-1}\int_{\mathbb{R}} \omega e^{-\frac{v_S}{2c_S^2}\omega^2\tau} e^{\sqrt{-1}\omega(\tau-t)}\, d\omega = -\frac{\sqrt{2\pi}c_S}{\sqrt{v_S\tau}} \frac{\partial}{\partial t} e^{-\frac{c_S^2(\tau-t)^2}{2v_S\tau}},$$

it follows that

$$\tilde{L}[\phi](t) = \int_0^{+\infty} \frac{t}{\tau}\phi(\tau)\frac{c_S}{\sqrt{2\pi v_S \tau}} e^{-\frac{c_S^2(\tau-t)^2}{2v_S\tau}}\, d\tau. \tag{29}$$

Analogously,

$$\tilde{L}^*[\phi](t) = \int_0^{+\infty} \frac{\tau}{t}\phi(\tau)\frac{c_S}{\sqrt{2\pi v_S t}} e^{-\frac{c_S^2(\tau-t)^2}{2v_S t}}\, d\tau. \tag{30}$$

Since the phase in (30) is quadratic and v_S is small then by consequence of the stationary phase Theorem A1, we have following result:

Theorem 4.1 (*Approximation of operator L*)

$$\tilde{L}^*[\phi] = \phi + \frac{v_S}{2c_S^2}\partial_{tt}(t\phi) + o\left(\frac{v_S}{c_S^2}\right), \quad \tilde{L}[\phi] = \phi + \frac{v_S}{2c_S^2} t\partial_{tt}\phi + o\left(\frac{v_S}{c_S^2}\right), \tag{31}$$

and therefore

$$\tilde{L}^*\tilde{L}[\phi] = \phi + \frac{v_S}{c_S^2}\partial_t(t\partial_t\phi) + o\left(\frac{v_S}{c_S^2}\right), \tag{32}$$

and,

$$(L^*\tilde{L})^{-1}[\phi] = \phi - \frac{v_S}{c_S^2}\partial_t(t\partial_t\phi) + o\left(\frac{v_S}{c_S^2}\right). \tag{33}$$

Proof

1. *Proof of approximation* (*31*):
   Let us first consider the case of operator L^*. We have

$$\tilde{L}^*[\phi](t) = \int_0^{+\infty} \frac{\tau}{t}\phi(\tau)\frac{c_S}{\sqrt{2\pi v_S t}} e^{-\frac{c_S^2(\tau-t)^2}{2v_S t}}\, d\tau = \frac{1}{t\sqrt{\varepsilon}}\left(\int_0^{+\infty} \psi(\tau) e^{if(\tau)/\varepsilon}\right),$$

   with, $f(\tau) = i\pi(\tau-t)^2$, $\varepsilon = 2\pi v_S t/c_S^2$ and $\psi(\tau) = \tau\phi(\tau)$. Remark that the phase f satisfies at $\tau = t$, $f(t) = 0$, $f'(t) = 0$, $f''(t) = 2i\pi \neq 0$. Moreover, we have

$$\begin{cases} e^{if(t)/\varepsilon}(\varepsilon^{-1}f''(t)/2i\pi)^{-1/2} = \sqrt{\varepsilon} \\ g_t(\tau) = f(\tau) - f(t) - \frac{1}{2}f''(t)(\tau-t)^2 = 0 \\ L_1[\psi](t) = L_1^{(1)}[\psi](t) = \frac{-1}{2i}f''(t)^{-1}\psi''(t) = \frac{1}{4\pi}(t\phi)''. \end{cases}$$

   Thus, Theorem A1 implies that

$$\left|\tilde{L}^*[\phi](t) - \left(\phi(t) + \frac{v_S}{2c_S^2}(t\phi)''\right)\right| \leqslant \frac{C}{t}\varepsilon^{3/2}\sum_{\alpha\leqslant 4}\sup|(t\phi)^{(\alpha)}|.$$

   The case of the operator $\tilde{L}$ is very similar. Note that

$$\tilde{L}[\phi](t) = \int_0^{+\infty} \frac{t}{\tau}\phi(\tau)\frac{c_S}{\sqrt{2\pi v_S \tau}} e^{-\frac{c_S^2(\tau-t)^2}{2v_S\tau}}\, d\tau = \frac{t}{\sqrt{\varepsilon}}\left(\int_0^{+\infty} \psi(\tau) e^{if(\tau)/\varepsilon}\right),$$

with $f(\tau)=\mathrm{i}\pi(\tau-t)^2/\tau$, $\varepsilon=\nu_s/2\pi c_s^2$ and $\psi(\tau)=\phi(\tau)\tau^{-3/2}$. It follows that

$$f'(\tau)=\mathrm{i}\pi\left(1-\frac{t^2}{\tau^2}\right),\quad f''(\tau)=2\mathrm{i}\pi\frac{t^2}{\tau^3},\quad f''(t)=2\mathrm{i}\pi\frac{1}{t},$$

and the function $g_t(\tau)$ is equal to

$$g_t(\tau)=\mathrm{i}\pi\frac{(\tau-t)^2}{\tau}-\mathrm{i}\pi\frac{(\tau-t)^2}{t}=\mathrm{i}\pi\frac{(t-\tau)^3}{\tau t}.$$

We deduce that

$$\begin{cases}(g_t\psi)^{(4)}(t) = (g_t^{(4)}(t)\psi(t)+4g_t^{(3)}(t)\psi'(t))=\mathrm{i}\pi\left(\frac{24}{t^3}\psi(t)-\frac{24}{t^2}\psi'(t)\right)\\ (g_t^2\psi)^{(6)}(t) = (g_t^2)^{(6)}(t)\psi(t)=-\pi^2\frac{6!}{t^4}\psi(t),\end{cases}$$

and then,

$$\begin{cases}L_1^{(1)}[\psi](t) = \frac{-1}{i}\left(\frac{1}{2}(f''(t))^{-1}\psi''(t)\right)=\frac{1}{4\pi}t\left(\frac{\tilde{\phi}}{\sqrt{t}}\right)''=\frac{1}{4\pi}\left(\sqrt{t}\tilde{\phi}''(t)-\frac{\tilde{\phi}'(t)}{\sqrt{t}}+\frac{3}{4}\frac{\tilde{\phi}}{t^{3/2}}\right)\\ L_1^{(2)}[\psi](t) = \frac{1}{8i}f''(t)^{-2}(g_t^{(4)}(s)\psi(s)+4g_t^{(3)}(t)\psi'(t))=\frac{1}{4\pi}\left(3\left(\frac{\tilde{\phi}(t)}{\sqrt{t}}\right)'-3\frac{\tilde{\phi}(t)}{t^{3/2}}\right)\\ \qquad = \frac{1}{4\pi}\left(3\frac{\tilde{\phi}'(t)}{\sqrt{t}}-\frac{9}{2}\frac{\tilde{\phi}(t)}{t^{3/2}}\right)\\ L_1^{(3)}[\psi](t) = \frac{-1}{2^3 2!3!i}f''(t)^{-3}(g_t^2)^{(6)}(t)\psi(s)=\frac{1}{4\pi}\left(\frac{15}{4}\frac{\tilde{\phi}(t)}{t^{3/2}}\right),\end{cases}$$

where $\tilde{\phi}(\tau)=\phi(\tau)/\tau$. Therefore, we have

$$\begin{aligned}L_1[\psi](t)&=L_1^{(1)}[\psi](t)+L_1^{(2)}[\psi](t)+L_1^{(3)}[\psi](t)\\ &=\frac{1}{4\pi}\left(\sqrt{t}\tilde{\phi}''(t)+(3-1)\frac{\tilde{\phi}'(t)}{\sqrt{t}}+\left(\frac{3}{4}-\frac{9}{2}+\frac{15}{4}\right)\frac{\tilde{\phi}(t)}{t^{3/2}}\right)=\frac{1}{4\pi\sqrt{t}}(t\tilde{\phi}(t))''=\frac{1}{4\pi\sqrt{t}}\phi''(t),\end{aligned}$$

and again Theorem A1 shows that

$$\left|\tilde{L}[\phi](t)-\left(\phi(t)+\frac{\nu_s}{2c_s^2}t\phi''(t)\right)\right|\leqslant Ct\varepsilon^{3/2}\sum_{\alpha\leqslant 4}\sup|\psi^{(\alpha)}(t)|.$$

2. *Proof of approximation* (32):
   Approximation (32) is evident and directly comes from (31).
3. *Proof of approximation* (33):
   Note that $\psi=(L^*\tilde{L})^{-1}[\phi]$ implies $(L^*\tilde{L})[\psi]=\phi$. As $(\nu_s/c_s^2)\ll 1$, we introduce the following asymptotic development of ψ:

$$\psi=\sum_{i=0}^{\infty}\left(\frac{\nu_s}{c_s^2}\right)^i\psi_i.$$

From (32), it holds

$$\psi_0+\left(\frac{\nu_s}{c_s^2}\right)((t\psi_0')'+\psi_1)+o\left(\frac{\nu_s}{c_s^2}\right)=\phi,$$

and

$$\psi_0=\phi\quad\text{and}\quad\psi_1=-\partial_t(t\partial_t\psi_0)=-\partial_t(t\partial_t\phi),$$

and finally

$$(L^*\tilde{L})^{-1}[\phi]=\phi-\frac{\nu_s}{c_s^2}\partial_t(t\partial_t\phi)+o\left(\frac{\nu_s}{c_s^2}\right).$$

Remark 4.2 (*Approximation of L for fractional models*)
For more general media with fractional power-law exponents γ, one can recover the ideal Green function from the viscous one in a very similar fashion by inverting a fractional differential operator. Such an approximation has been reported in [15], in the context of Photoacoustic imaging in dissipative media. See [15, Section 1.2.6] for a brief account of the approximation for fractional model.

Remark 4.3 (*Imaging procedure*)
From Theorem 4.1, it follows that the ideal Green function, $\delta(\tau - r/c_s)/(4\pi r)$, can be approximately reconstructed from the viscous Green function, $G^s(r,t;\xi)$, by either solving the ordinary differential equation

$$\phi + \frac{\nu_s}{c_s^2}\partial_t(t\partial_t\phi) = L^*[G^s](r,t;\xi),$$

with $\phi = 0$, $t \ll 0$ or just making the approximation

$$\frac{1}{(4\pi r)}\delta(\tau - r/c_s) \approx L^*[G^s](r,t;\xi) - \frac{\nu_s}{c_s^2}\partial_t(t\partial_t[L^*G^s](r,t;\xi)).$$

Once the ideal Green function $\delta(\tau - r/c_s)/(4\pi r)$ is reconstructed, one can find its source ξ using a time-reversal, a Kirchhoff or a back propagation algorithm. See [5–8].

Using the asymptotic formalism developed in [8, 16, 17], one can also find the shear modulus of the anomaly using the ideal near-field measurements which can be reconstructed from the near-field measurements in the viscous medium. The asymptotic formalism reduces the anomaly imaging problem to the detection of the location and the reconstruction of a certain polarization tensor in the far-field and separates the scales in the near-field. □

5. Numerical illustrations

5.1. Profile of the green function

In this section, we illustrate the profile of the Green function for different values of the power law exponent γ, shear viscosity η_s and t. We choose parameters of simulation as in the work of Bercoff *et al.* [1]: we take $\rho = 1{,}000$, $c_s = 1$, $c_p = 40$, $\eta_p = 0$.

In Figure 2, we plot the first component, G_{11}, of the Green function observed at the point $A = ((1/\sqrt{2})r, (1/\sqrt{2})r, 0)$ (see first image in Figure 1) with $r = 0.015$ for three different values of γ and two different values of η_s. We can clearly distinguish the three different

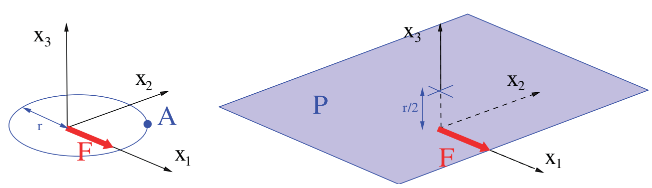

Figure 1. Left: $A = \left(\frac{1}{\sqrt{2}}r, \frac{1}{\sqrt{2}}r, 0\right)$. Right: Plane $P = \{x \in \mathbb{R}^3, x_3 = \frac{r}{2}\}$.

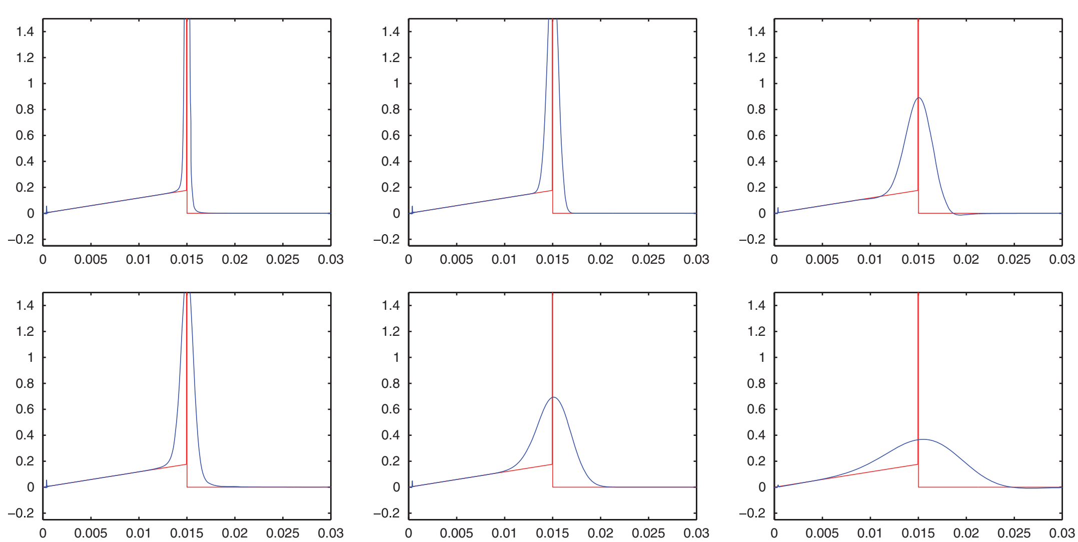

Figure 2. Temporal response $t \to G_{11}(A,t,0)$ to a spatiotemporal delta function using a purely elastic Green function and a viscous Green function (blue line): First line : $\eta_s = 0.02$, Second line : $\eta_s = 0.2$; (left to right) $\gamma = 1.75$, $\gamma = 2$, $\gamma = 2.25$.

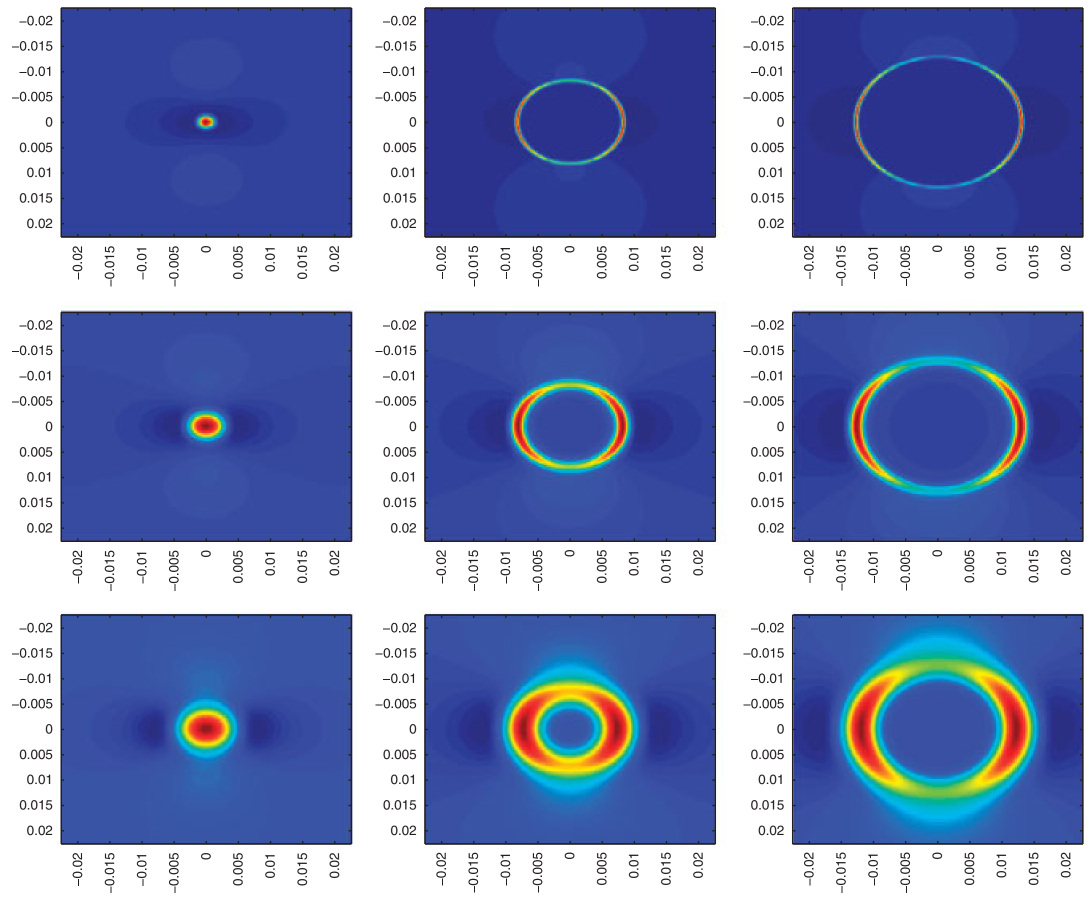

Figure 3. 2D spatial response $x \to G_{11}(x,t,0)$ on the plan P to a spatiotemporal delta function with (top to bottom): a purely elastic Green function, a viscous Green function with $(\gamma=1.75, \eta_s=0.2)$ and $(\gamma=2, \eta_s=0.2)$. Left to right: $t=0.0075$, $t=0.0112$ and $t=0.015$.

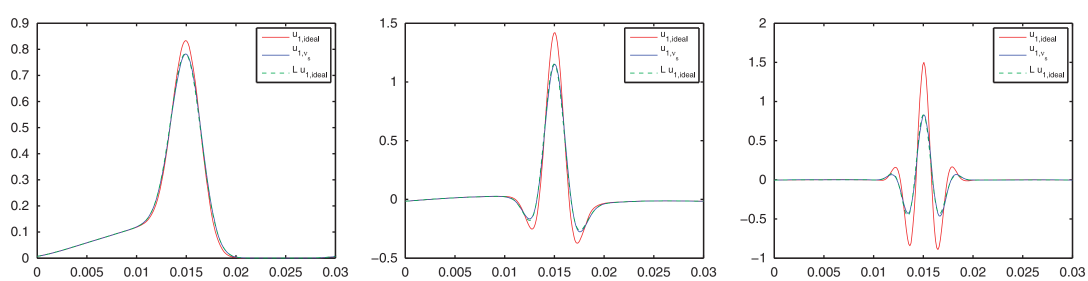

Figure 4. Comparison between $u_{1,\nu_s}(x,t)$ and $L[u_{1,\text{ideal}}](x,t)$ observed at $x=A$ with $\gamma=2$ and $\eta_s=0.02$; Left; $\omega_0=0$; Center, $\omega_0=\rho$; Right, $\omega_0=2\rho$.

terms of the Green function; i.e. G^s_{ij}, G^p_{ij} and G^{ps}_{ij} and that the attenuation behavior varies with respect to different choices of power law exponent γ and the viscosity η_s.

In Figure 3, we plot G_{11}, evaluated on the plane $P=\{x\in\mathbb{R}^3; x_3=r/2\}$ (see second image in Figure 1), at three different times. As expected, we get a diffusion of the wavefront with the increasing values of the power law exponent γ and depending on the choice of η_s.

5.2. Approximation of attenuation operator L

Consider the limiting case when $\lambda\to+\infty$ with $\gamma=2$. We take $\rho=1,000$, $c_s=1$ and a concentrated force **F**. of the form $\mathbf{F}=-T(t)\delta(x)\mathbf{e}_1$ where the time profile of the pulse, $T(t)$, is a Gaussian with central frequency ω_0 and bandwidth ρ. Denote by $\vec{u}_{\text{ideal}}(x,t)$ the ideal response without attenuation and by $\vec{u}_{\nu_s}(x,t)$, the response associate to the attenuation coefficient ν_s. Following Section 4, we have

$$\vec{u}_{\nu_s} \approx L[\vec{u}_{\text{ideal}}].$$

In Figure 4, we plot the first components of $t\to\vec{u}_{\text{ideal}}(A,t)$, $t\to\vec{u}_{\nu_s}(A,t)$ and $t\to L[\vec{u}_{\text{ideal}}](A,t)$ for different values of ω_0 and $\eta_s=0.02$. As expected, the function $t\to\vec{u}_{\nu_s}(A,t)$ and $t\to L[\vec{u}_{\text{ideal}}](A,t)$ are very similar. It justifies that the attenuation operator, L, well describes the viscosity effects and the approximations presented in Section 4 are quite adequate.

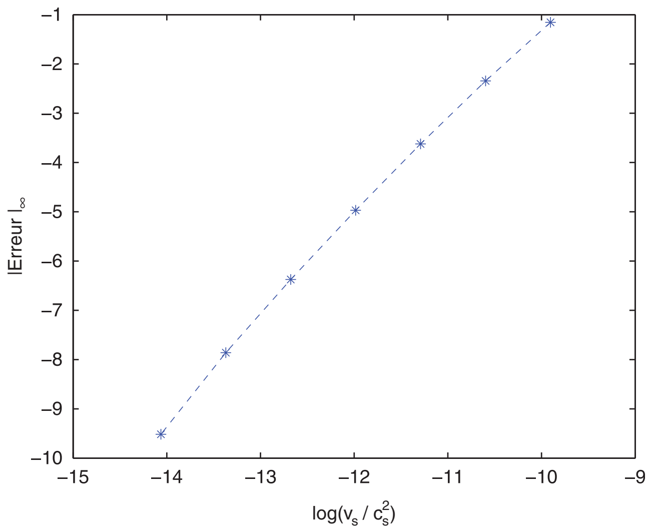

Figure 5. Approximation of operator L : Error $\frac{v_s}{c_s^2}\rightarrow\left\|L[\phi]-\left(\phi+\frac{v_s}{2c_s^2}t\phi''\right)\right\|_\infty$ in logarithmic scale in the case when $\phi(t)=u_{1,\text{ideal}}(A,t)$ with $\omega_0=\rho$.

Finally, in Figure 5, we plot in logarithmic scale the error of approximation

$$\frac{v_S}{c_S^2}\rightarrow\left\|L[\phi]-\left(\phi+\frac{v_S}{2c_S^2}t\phi''\right)\right\|_\infty,$$

where $\phi(t)$ is the first component of $\vec{u}_{\text{ideal}}(x,t)$, computed at the point $x=A$ with $\omega_0=\rho$. It clearly appears to be an approximation of order 2.

6. Conclusion

In this paper, we have computed the Green function in a visco-elastic medium obeying a frequency power-law. For the Voigt model, which corresponds to a quadratic frequency loss, we have used the stationary phase Theorem A1 to reconstruct the ideal Green function from the viscous one by solving an ordinary differential equation. Once the ideal Green function is reconstructed, one can find its source point ξ using the algorithms in [5–8] such as time reversal, back-propagation, and Kirchhoff imaging. For more general power-law media, one can recover the ideal Green function from the viscous one by inverting a fractional differential operator [15].

A number of recent experimental studies indicate that certain tissues like muscles and glands exhibit anisotropic visco-elastic behavior (see e.g. [18–20]). Therefore, it would be very interesting to approximate the ideal Green function from the viscous one in an anisotropic medium and would be the subject of future investigations.

Appendix A: Stationary phase method

The proof of the following theorem is established in [21, Theorem 7.7.1].

Theorem A1 (*Stationary Phase*)
Let $K\subset[0,\infty)$ be a compact set, X an open neighborhood of K and k a positive integer. If $\psi\in C_0^{2k}(K)$, $f\in C^{3k+1}(X)$ and $Im(f)\geqslant 0$ in X, $Im(f(t_0))=0$, $f'(t_0)=0$, $f''(t_0)\neq 0$, $f'\neq 0$ in $K\setminus\{t_0\}$ then for $\varepsilon>0$

$$\left|\int_K\psi(t)e^{if(t)/\varepsilon}dt-e^{if(t_0)/\varepsilon}\left(\lambda f''(t_0)/2\pi i\right)^{-1/2}\sum_{j<k}\varepsilon^j L_j[\psi]\right|\leqslant C\varepsilon^k\sum_{\alpha\leqslant 2k}\sup|\psi^{(\alpha)}(x)|.$$

Here C is bounded when f stays in a bounded set in $C^{3k+1}(X)$ and $|t-t_0|/|f'(t)|$ has a uniform bound. With,

$$g_{t_0}(t)=f(t)-f(t_0)-\tfrac{1}{2}f''(t_0)(t-t_0)^2,$$

which vanishes up to third order at t_0, we have

$$L_j[\psi]=\sum_{\nu-\mu=j}\sum_{2\nu\geqslant 3\mu}i^{-j}\frac{2^{-\nu}}{\nu!\,\mu!}(-1)^\nu f''(t_0)^{-\nu}(g_{t_0}^\mu\psi)^{(2\nu)}(t_0).$$

Note that L_1 can be expressed as the sum $L_1[\psi]=L_1^{(1)}[\psi]+L_1^{(2)}[\psi]+L_1^{(3)}[\psi]$, where L_1^j, for $j=1,2,3$ are, respectively, associate to the pair $(\nu_j,\mu_j)=(1,0),(2,1),(3,2)$ and are identified as:

$$\begin{cases} L_1^{(1)}[\psi] = \dfrac{-1}{2i} f''(t_0)^{-1}\psi^{(2)}(t_0), \\ L_1^{(2)}[\psi] = \dfrac{1}{2^2 2!\, i} f''(t_0)^{-2}(g_{t_0}u)^{(4)}(t_0) = \dfrac{1}{8i} f''(t_0)^{-2}(g_{t_0}^{(4)}(t_0)\psi(t_0)+4g_{t_0}^{(3)}(t_0)\psi'(t_0)), \\ L_1^{(3)}[\psi] = \dfrac{-1}{2^3 2!\,3!\, i} f''(t_0)^{-3}(g_{t_0}^2\psi)^{(6)}(t_0) = \dfrac{-1}{2^3 2!\,3!\, i} f''(t_0)^{-3}(g_{t_0}^2)^{(6)}(t_0)\psi(t_0). \end{cases}$$

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