School of Natural Sciences
Department of Mathematics
Quiz 1 (Integral Equations), Spring 2020
Solution-Key

NUST
Defining futures

AQ. Find the characteristic values and associated non-trivial solutions (if any) of the equation

$$h(x) = \lambda \int_0^{2\pi} \sin x \cos y \, h(y) dy. \tag{1}$$

Sol. The solution to (1) is given by $h(x) = C\lambda \sin x$ with $C := \int_0^{2\pi} \cos y \, h(y) dy$. In order to find C, multiply Eq. (1) with $\cos x$ and integrate over $[0, 2\pi]$, i.e.,

$$C = \int_0^{2\pi} \cos y h(y) dy = \lambda \int_0^{2\pi} \sin x \cos x \, dx \int_0^{2\pi} \cos y h(y) dy = C\lambda \int_0^{2\pi} \sin x \cos x \, dx.$$

On simplification, we arrive at

$$C \left[1 - \lambda \int_0^{2\pi} \sin x \, \cos x \, dx \right] = 0. \tag{a}$$

The non-trivial solutions to Eq. (1) exist only if

$$1 - \lambda \int_0^{2\pi} \sin x \, \cos x \, dx = 0. \tag{b}$$

However, this is absurd because

$$\int_0^{2\pi} \sin x \cos x \, dx = \frac{1}{2} \int_0^{2\pi} \sin 2x \, dx = -\frac{1}{4} \left[\cos 2x \Big|_0^{2\pi} = 0.$$

This will lead to $1 = 0$ in Eq. (b), which is absurd. Therefore, the only solution to (a) is $C = 0$. Thus, there is only a trivial solution to (1) and there are no characteristic values.

RQ Consider the integral equations

$$t = \int_0^t \left(e^t + e^x \right) \varphi(x) dx. \tag{2}$$

(a) Tick the appropriate option in each case (only one). Integral equation (2) is:

1. **Volterra** ✓   Fredholm
2. **Linear** ✓   Non-Linear
3. Homogeneous   **Non-Homogeneous** ✓
4. Singular   **Non-Singular** ✓

(b) Transform the first kind integral equation (2) to a second kind integral equation.

Sol. Differentiating Eq. (2) with respect to t, we get

$$1 = \frac{dt}{dt}(e^t + e^t)\varphi(t) - 0 + \int_0^t \frac{\partial}{\partial t}\left(e^t + e^x\right)\varphi(x)dx = 2e^t\varphi(t) + \int_0^t e^t\varphi(x)dx.$$

On rearranging, we get

$$2e^t\varphi(t) = 1 - \int_0^t e^t\varphi(x)dx,$$

or equivalently,

$$\varphi(t) = \frac{1}{2}e^{-t} - \frac{1}{2}\int_0^t \varphi(x)dx.$$

"If you believe it will work out, you'll see opportunities. If you believe it won't, you will see obstacles." — Wayne Dyer.