School of Natural Sciences
Department of Mathematics
Quiz 2 (Integral Equations), Spring 2020
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AQ. Find the solution y to the boundary value problem

$$\begin{cases} y'' = x^2, & x \in (0,1), \\ y(0) = 0 = y(1), \end{cases}$$

using the associated **Green's function**

$$G(x,s) = \begin{cases} s(x-1), & 0 \le s \le x, \\ x(s-1), & x \le s \le 1. \end{cases}$$

Sol. Note that $f(x) = x^2$ here and the Green's function satisfies the boundary conditions $G(0,s) = 0 = G(1,x)$. The solution of the given boundary value problem is

$$\begin{aligned} y(x) &= \int_0^1 G(x,s)f(s)ds \\ &= \int_0^1 G(x,s)s^2 ds \\ &= \int_0^x s(x-1)s^2 ds + \int_x^1 x(s-1)s^2 ds \\ &= (x-1)\int_0^x s^3 ds + x\int_x^1 (s^3 - s^2)ds \\ &= (x-1)\left[\frac{s^4}{4}\right]_0^x + x\left[\frac{s^4}{4} - \frac{s^3}{3}\right]_x^1 \\ &= (x-1)\frac{x^4}{4} + x\left(\frac{1}{4} - \frac{1}{3}\right) - x\left(\frac{x^4}{4} - \frac{x^3}{3}\right) \\ &= \frac{1}{12}\left(x^4 - x\right). \end{aligned}$$

RQ. Solve the Volterra equation

$$u(x) = 1 + \int_0^x (x-t)u(t)dt, \qquad \text{(a)}$$

by finding the solution of the equivalent ODE.

Sol. Differentiating (a) w.r.t. x, we get

$$u'(x) = \int_0^x u(t)dt.$$

Differentiating once again we get

$$u''(x) = u(x).$$

Moreover, $u(0) = 1$ and $u'(0) = 0$. Thus, we get boundary value problem

$$\begin{cases} u''(x) - u(x) = 0 \\ u(0) = 1, \quad u'(0) = 0. \end{cases} \tag{1}$$

The characteristic equation of the homogeneous ODE is $m^2 - 1 = 0$. Thus, $m = \pm 1$ and the general solution is $u(x) = c_1 e^x + c_2 e^{-x}$. Invoking the conditions $u(0) = 1$ and $u'(0) = 0$ we get, respectively,

$$1 = c_1 + c_2 \quad \text{and} \quad 0 = c_1 - c_2.$$

Thus, $c_1 = c_2 = 1/2$. Therefore, $u(x) = \dfrac{e^x + e^{-x}}{2} = \cosh x$.

"Learning is never done without errors and defeat." — Vladimir Lenin