National University of Technology, Islamabad
Quiz I-Key(Linear Algebra & ODE), Fall 2019
BET (Mechanical), Date: September 26, 2019

Q **Balancing a chemical equation** $wC_3H_8 + xO_2 \to yCO_2 + zH_2O$ means finding integers w, x, y, z such that the numbers of atoms of carbon (C), hydrogen (H) and oxygen (O) are the same on both sides of this reaction, in which propane C_3H_8 and O_2 give carbon dioxide and water. Find the **smallest positive integers** w, x, y and z using Echelon or Reduced Echelon Form. *(Hint: Compare the atoms of C, H and O on both sides).*

Sol. On comparing the atoms of Carbon (C), Hydrogen (H), and Oxygen (O) on both sides of the reaction equation, we get the following set of equations

$$3w = y; \qquad 8w = 2z; \qquad 2x = 2y + z,$$

or equivalently the system of equations

$$\begin{aligned} 3w - y &= 0, && (1)\\ 8w - 2z &= 0, && (2)\\ 2x - 2y - z &= 0. && (3) \end{aligned}$$

In matrix form, we have

$$\begin{pmatrix} 3 & 0 & -1 & 0 \\ 8 & 0 & 0 & -2 \\ 0 & 2 & -2 & -1 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{4}$$

We consider the augmented matrix $[A|b]$ and reduce it to an echelon or reduced echelon form as follows.

$$[A|b] = \begin{pmatrix} 3 & 0 & -1 & 0 & 0 \\ 8 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1/3 & 0 & 0 \\ 1 & 0 & 0 & -1/4 & 0 \\ 0 & 1 & -1 & -1/2 & 0 \end{pmatrix} \quad (\text{R.O.:} 1/3R_1;\ 1/8R_2;\ 1/2R_3)$$

$$\sim \begin{pmatrix} 1 & 0 & -1/3 & 0 & 0 \\ 0 & 0 & 1/3 & -1/4 & 0 \\ 0 & 1 & -1 & -1/2 & 0 \end{pmatrix} \quad (\text{R.O.:} R_2 - R1)$$

$$\sim \begin{pmatrix} 1 & 0 & -1/3 & 0 & 0 \\ 0 & 1 & -1 & -1/2 & 0 \\ 0 & 0 & 1/3 & -1/4 & 0 \end{pmatrix} \quad (\text{R.O.:} R_{23})$$

$$\sim \begin{pmatrix} 1 & 0 & -1/3 & 0 & 0 \\ 0 & 1 & -1 & -1/2 & 0 \\ 0 & 0 & 1 & -3/4 & 0 \end{pmatrix} \quad (\text{R.O.:} 3R_3)$$

This gives the echelon form of the augmented matrix. In order to find the reduced echelon matrix, we proceed as follows:

$$
\begin{aligned}
[A|b] &\sim \begin{pmatrix} 1 & 0 & -1/3 & 0 & 0 \\ 0 & 1 & -1 & -1/2 & 0 \\ 0 & 0 & 1 & -3/4 & 0 \end{pmatrix} \\
&\sim \begin{pmatrix} 1 & 0 & -1/3 & 0 & 0 \\ 0 & 1 & 0 & -5/4 & 0 \\ 0 & 0 & 1 & -3/4 & 0 \end{pmatrix} \quad (\text{R.O.:} R_2 + R_3) \\
&\sim \begin{pmatrix} 1 & 0 & 0 & -1/4 & 0 \\ 0 & 1 & 0 & -5/4 & 0 \\ 0 & 0 & 1 & -3/4 & 0 \end{pmatrix} \quad (\text{R.O.:} R_1 + 1/3R3)
\end{aligned}
$$

Therefore, the reduced equations are

$$w - (1/4)z = 0; \qquad x - (5/4)z = 0; \qquad y - (3/4)z = 0, \qquad z = \text{free variable.} \tag{5}$$

We choose $z = 4r$, for $r \in \mathbb{R}$ to get

$$w = r, \qquad x = 5r, \qquad y = 3r, \qquad z = 4r. \tag{6}$$

Note that the smallest positive integer solution $(w, x, y, z)^T$ is possible when $r = 1$. Thus, the balanced chemical equation is $C_3H8 + 5O_2 \rightarrow 3CO_2 + 4H_2O$.