National University of Technology, Islamabad
Quiz-II (LA and ODE), Fall 2019
Solution-Key

Q.1 Find the values of $\gamma \in \mathbb{R}$ for which the system of equations

$$\begin{cases} x - 2\gamma y = 1 \\ 2\gamma x - y = -1 \end{cases} \tag{1}$$

is inconsistent, is consistent or has unique solution.

Ans. Note that the augmented matrix corresponding to the system is

$$[A|b] := \begin{pmatrix} 1 & -2\gamma & 1 \\ 2\gamma & -1 & -1 \end{pmatrix},$$

that has the echelon form

$$\begin{aligned} [A|b] &\sim \begin{pmatrix} 1 & -2\gamma & 1 \\ 0 & -1+4\gamma^2 & -1-2\gamma \end{pmatrix} && (R_2 \to R_2 - 2\gamma R_1) \\ &\sim \begin{pmatrix} 1 & -2\gamma & 1 \\ 0 & 1-4\gamma^2 & 1+2\gamma \end{pmatrix} && (R_2 \to (-1)R_2). \end{aligned}$$

The system is inconsistent if $\text{Rank}(A) < \text{Rank}([A|b])$, i.e., $1 - 4\gamma^2 = 0$ but $1 + 2\gamma \neq 0$. This is only possible if $\gamma = 1/2$. Hence, for $\gamma = 1/2$ the system does not possess any solution.

The system is consistent if $\text{Rank}(A) = \text{Rank}([A|b])$, i.e., either $1 - 4\gamma^2$ and $1 + 2\gamma$ are both non-zero or they are both zero. When $1 - 4\gamma^2 = 0 = 1 + 2\gamma$, i.e., when $\gamma = -1/2$, the coefficient matrix as well as the augmented matrix, both are rank deficient, i.e., there are two unknowns but one independent equation, i.e., there is one free variable. So, in that case there are infinite many solutions. If $1 - 4\gamma^2 \neq 0$ and $1 + 2\gamma \neq 0$, i.e. $\gamma \neq \pm 1/2$ then $\text{Rank}(A) = 2 = \text{Rank}([A|b])$. Therefore, the system is consistent with no free variables. Thus, there is a unique solution for each $\gamma \in \mathbb{R}$ such that $\gamma \neq \pm 1/2$.

"Every stumble is not a fall, and every fall does not mean failure." ~ Oprah Winfrey