NATIONAL UNIVERSITY OF TECHNOLOGY, ISLAMABAD
QUIZ III (LINEAR ALGEBRA AND ODE), SPRING 2019
DATED: OCTOBER 21, 2019

Q.1 Let transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ be given by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} := T\begin{pmatrix} x \\ 0 \\ y \\ 0 \end{pmatrix}.$$

Find $\ker(T)$ and $\dim(\ker(T))$. Without calculating $\text{rang}(T)$, precise the $\dim(\text{rang}(T))$.

Ans. It is evident that T is a linear transformation with matrix $A := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Therefore, for all $\mathbf{x} \in \mathbb{R}^3$, we have $T(\mathbf{x}) = A\mathbf{x}$. Thus, the $\ker(T) = \text{null}(A)$ is the set of all solutions to the homogeneous linear system $A\mathbf{x} = \mathbf{0}$. Note that

$$A \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad (R_{23}).$$

Therefore, $x = 0$, $y = 0$, and $z =$free variable, or equivalently,

$$\ker(T) = \text{null}(A) = \left\{ \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; z \in \mathbb{R} \right\}.$$

It is evident that

$$\ker(T) = \left\{ z\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; z \in \mathbb{R} \right\} = \text{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} = \text{span}\{\mathbf{e}_1\}.$$

As $\{\mathbf{e}_1\}$ spans $\ker(T)$ and that $\{\mathbf{e}_1\}$ is linearly independent, it forms a basis of $\ker(T)$. Since there is only 1 element in the basis of $\ker(T)$, we have $\dim(\ker(T)) = 1$.

From Rank-Nullity Theorem, we have $\dim(\text{rang}(T)) = 3 - \dim(\ker(T)) = 2$.

"What seems to us as bitter trials are often blessings in disguise." ∼Oscar Wilde